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# GENERIC REPRESENTATIONS FOR THE UNITARY GROUP IN THREE VARIABLES

BY

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ABSTRACT

We show that for the quasi-split unitary group in three variables every tempered packet of cuspidal automorphic representations contains a globally generic representation.

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## 1. Introduction

Suppose that $G$ is a reductive group, defined and quasi-split over a number field $F$. Let $N$ be a maximal unipotent subgroup in $G$ and $\theta$ a character of $N(F_{\mathbb{A}})$ trivial on $N(F)$ and generic. Let $\pi$ be a cuspidal automorphic representation of $G(F_{\mathbb{A}})$ and $V$ the isotypic component of the space of cusp forms corresponding to $\pi$. If $\phi$ is a smooth vector in $V$ we set

$$\mathcal{W}(\phi) := \int_{N(F)\backslash N(F_{\mathbb{A}})} \phi(n)\overline{\theta}(n)dn.$$

Thus $\mathcal{W}$ is a continuous linear form transforming under $\theta$. We say that $\pi$ is **globally generic** with respect to $\theta$ if the linear form $\mathcal{W}$ is not identically zero. If $\mathcal{W}$ is non-zero, then, for each place $v$, the component $\pi_v$ is **generic** with respect to $\theta_v$, that is, admits a non-zero linear form $\mathcal{W}_v$ transforming under $\theta_v$ (**Whittaker** linear form). Recall that such a form is unique, within a scalar factor. If we define $W_\phi(g) = \int \phi(ng)\overline{\theta}(n)dn$ then the space $\mathcal{W}$ spanned by the functions $W_\phi$ is irreducible. The closure $V_1$ of the kernel of the map $\phi \mapsto W_\phi$ is invariant and so we may decompose $V$ into an orthogonal direct sum $V = V_1 \oplus V_0$. The map $\phi \mapsto W_\phi$ is now injective on the space of smooth vectors of $V_0$. It follows that the representation on $V_0$ is irreducible. It is generally conjectured that inside a tempered $L$-packet of automorphic representations, there is exactly one component which is generic with respect to a given generic character $\theta$ (cf. [KRS] and [fS2]). The purpose of this paper is to establish the conjecture for the quasi-split group in three variables relative to a quadratic extension $E/F$. We note that in general we may have $V \neq V_0$. However, here $V = V_0$, that is, the representation $\pi$ has multiplicity one ([Ro1], Theorem 13.3.1, p. 201). Before explaining the proof, we also raise another, related, question. If $\pi$ is as above, we can define a global **Bessel** distribution as follows:

$$J_\pi(f) := \sum_\phi \mathcal{W}(\rho(f)\phi)\overline{\mathcal{W}(\phi)};$$

here the sum is over an orthonormal basis of $V_0$; it amounts to the same to take the sum over a basis of $V$, or even a basis of the space of the packet to which $\pi$ belongs—if the conjecture is true. At a place $v$ we may define a local **Bessel** distribution

$$\mathcal{B}_v(f_v) := \sum_\phi \mathcal{W}_v(\pi(f_v)\phi)\overline{\mathcal{W}_v(\phi)}.$$

Note that the distribution is defined within a positive factor. Thus, to make the following statement precise, one would need to choose a normalization of this distribution. It follows from the local uniqueness that the global Bessel distribution decomposes as an infinite product of the local Bessel distributions:

$$J_\pi(f) = C \prod_v \mathcal{B}_v(f_v), \quad \text{if } f = \otimes f_v. \tag{1}$$

The constant $C$ is positive as a ratio of positive type distributions. The question at hand is to compute the constant in terms of $L$-functions attached to $\pi$. In the case of the unitary group, it is clear that with more local information we could answer this question completely. See [BM] for another example.

The results of the present paper were announced in a short note with Solomon Friedberg as a coauthor ([FGJR]). It is a pleasure to acknowledge his contribution to the genesis of this work.

We now proceed to discuss the proof of the result. We let $E/F$ be a quadratic extension of number fields with Galois group $\{1,\sigma\}$. We often write $\sigma(z)=\overline{z}$. We let $U_1$ be the unitary group in one variable, that is, the group of elements of norm 1 in $E^\times$. We denote by $G$ the group $GL(3)$ regarded as an algebraic group over $E$ and we denote by $Z$ its center. We let $H$ be the group $GL(3,F)$ regarded as an algebraic group over $F$ and we denote by $Z_H$ its center. To prove the result, we use the concept of a cuspidal distinguished representation of $G(E_{\mathbb{A}})$. In a precise way, we say that such a $\Pi$ is **distinguished** by $H(F_{\mathbb{A}})$ (or simply distinguished), if the central character $\omega$ of $\Pi$ is trivial on $F_{\mathbb{A}}^\times$ (that is, is distinguished), and there is a form $\phi$ in the space of $\Pi$ such that

$$\mathcal{I}(\phi):=\int_{Z_H(\mathbb{A})H(F)\backslash H(F_{\mathbb{A}})}\phi(h)dh$$

is non-zero. On the one hand, it is a result of [F2], that, if $\Pi$ is distinguished, then $\Pi^\sigma=\tilde{\Pi}$, where $\Pi^\sigma$ is the representation defined by $\Pi^\sigma(g)=\Pi(g^\sigma)$ and $\tilde{\Pi}$ denotes the representation contragredient to $\Pi$. On the other hand, it is a result of [F2] and [FZ], that $\Pi$ is distinguished if and only if the (partial) Asai $L$-function attached to $\Pi$ has a pole at $s=1$. From this and the factorization of the Rankin–Selberg $L$-function attached to the pair $(\Pi,\Pi^\sigma)$ in terms of Asai $L$-functions, it follows that if the central character of $\Pi$ is distinguished and $\Pi^\sigma=\tilde{\Pi}$, then $\Pi$ is distinguished (section 2). In turn, this condition of symmetry is equivalent to $\Pi$ being the standard (or stable) base change of a tempered stable packet of automorphic representations of the unitary group. Thus for $GL(3)$ at least, the property of $\Pi$ being distinguished is equivalent to $\Pi$ being the functorial image of a stable packet of automorphic cuspidal representations. This fact is predicated by the analysis of the potential pole of the Asai $L$-function in terms of the $L$-group ([F2]).

Our main tool is then the relative trace formula. The relative trace formula used here was first discussed in [Y] in the context of $GL(2)$. In fact, the split unitary group in [Y] is replaced by the group $GL(2,F)$. However, the functions at hand are in fact supported on the subgroup $G'$ of matrices with determinant a norm, and the group $G'$ is (up to central tori) isomorphic to the unitary group in two variables. In [F2], this is exploited to reformulate (and extend) the formula of [Y] in terms of the unitary group in two variables. Reference [F3] contains a study in the context of $GL(n)$ of the formula used here in the context of

$GL(3)$. The motivation of [F3] is essentially the same as ours. Indeed, given the above discussion for $GL(3)$, the conjecture stated in [F2] (which motivates [F3]) amounts to saying that every stable tempered packet of representations of the unitary group contains a generic representation. We use the discussion of [F3]. However, the analysis of the continuous spectrum presented in [F3], although suggestive, is insufficient and erroneous. We do provide the analysis for $GL(3)$, which we hope is correct. It is based on the results of [JLR].

We now proceed to discuss the trace formula in question. Recall we denote by $G$ the group $GL(3)$ regarded as an algebraic group over $E$. We let $B$ be the group of upper triangular matrices, $A$ the group of diagonal matrices, $N$ the group of upper triangular matrices with unit diagonal, and $Z$ the group of scalar matrices in $G$. We write an element $g$ as $g = (g_{i,j})$, where $i$ is the column index (this is the opposite of the standard convention). We fix a non-trivial character $\psi$ of $F_{\mathbb{A}}/F$, a character $\zeta$ of $U_1(F_{\mathbb{A}})/U_1(F)$ and denote by $\omega$ the character of $E_{\mathbb{A}}^{\times}/E^{\times}$ defined by $\omega(z) = \zeta(z\overline{z}^{-1})$.

If $f$ is a smooth function of compact support on $GL(3, E_{\mathbb{A}})$ we define a kernel

$$K_f(x,y) = \int_{Z(E_{\mathbb{A}})/Z(E)} \left( \sum_{\xi \in GL(3,E)} f(x^{-1}\xi y z) \right) \omega(z) dz. \tag{2}$$

Our main object of study is the distribution

$$J(f) := \int_{H(F)Z_H(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} \int_{N(E)\backslash N(E_{\mathbb{A}})} K_f(h,n)\theta(n) dn dh. \tag{3}$$

Here $\theta$ is the character of $N(E_{\mathbb{A}})$ defined by

$$\theta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x-y), \quad \psi_E(z) = \psi(z+\overline{z}). \tag{4}$$

The distribution can be computed in terms of the symmetric space

$$\mathfrak{S} = \{s \in GL(3,E) \colon s\overline{s} = 1\}.$$

Indeed, we denote by $\mathcal{P}$ the polarization map

$$\mathcal{P} \colon G \to S, \quad \mathcal{P}(g) := \overline{g}^{-1} g.$$

The group $G(E)$ operates on the right by twisted conjugation on $\mathfrak{S}$:

$$s \overset{g}{\mapsto} \overline{g}^{-1} s g.$$

There is a smooth function of compact support $\Phi$ on $\mathfrak{S}(F_{\mathbb{A}})$ such that

$$\Phi(\mathcal{P}(g)) = \int_{H(F_{\mathbb{A}})} f(hg)dh.$$

Then

$$(5) \qquad J(f) = \int_{N(E)\backslash N(E_{\mathbb{A}})} \int_{\mathbb{U}_1(F)\backslash \mathbb{U}_1(F_{\mathbb{A}})} \left( \sum_{\xi \in \mathfrak{S}(F)} \Phi(\overline{n}^{-1}\xi n u) \right) \theta(n) dn \zeta(u) du.$$

In this paper, we will make the following simplifying assumption: we let $S_0$ be a finite set of places of $F$ and $S$ be the corresponding set of places of $E$. We assume that $S_0$ contains the places at infinity, the even finite places, the places which ramify in $E$, the places where $\psi$ ramifies and the places where $\zeta$ ramifies. We assume that $f$ is a product of local functions $f_v$. For $v \notin S$ we assume that $f_v$ is bi-invariant under the standard maximal compact subgroup $K_v$. Furthermore, we assume that for almost all such $v$'s, the function $f_v$ is the characteristic function of $K_v$. We make the further assumption that for all places $v \in S_0$ which are inert in $E$ and thus under one place $v$ of $E$, the function $f_v$ is supported on the open set $\Omega_v \subset GL(3, E_v)$ of matrices of the form $h\eta a n$ with $h \in H_v$, $a \in A_v$, $n \in N_v$. Here $\eta$ is an element such that $\mathcal{P}(\eta) = w$, and

$$(6) \qquad w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We do assume that there is at least one such place. The corresponding function $\Phi_v$, defined by

$$\Phi_v(\mathcal{P}(g)) = \int_{H_v} f_v(hg)dh,$$

is supported on the open set $\Omega^w_{v_0} := \mathcal{P}(\Omega_v)$. This is the set of matrices $s \in \mathfrak{S}_{v_0}$ such that $s_{1,3} \neq 0$. Every matrix $s$ in $\Omega^w_{v_0}$ can be written uniquely in the form

$$s = \overline{n}^{-1} w u d_a n$$

with $n \in N(E_v)$, $u$ a scalar matrix in $U_1(F_{v_0})$, and $d_a = \mathrm{diag}(a, 1, \overline{a}^{-1})$. The complement of $\Omega^w_{v_0}$ is the union of orbits of $N(E_v)$ with a non-trivial centralizer (cf. [F3]).

We consider the global orbital integrals, for $u \in U_1(F_{\mathbb{A}})$ and $a \in E^{\times}_{\mathbb{A}}$

$$(7) \qquad J(ud_a, \Phi) := \int_{N(E_{\mathbb{A}})} \Phi(\overline{n}^{-1} w u d_a n) \theta(n) dn.$$

Then

$$J(f)=\sum_{\alpha\in E^\times}\int_{U_1(F_{\mathbb{A}})}J(ud_\alpha,\Phi)\zeta(u)du.$$

There is also a notion of local orbital integrals. If $v_0$ is a place of $F$ inert in $E$ and $v$ the corresponding place of $E$, then the orbital integral is

$$J(ud_a,\Phi_{v_0}):=\int_{N(E_v)}\Phi_{v_0}(\overline{n}^{-1}wud_an)\theta(n)dn. \tag{8}$$

If $v_0$ splits into two places $v_1,v_2$ of $E$, then $\mathfrak{S}_{v_0}$ can be identified with the set of pairs $(g_1,g_2)$ with $g_1g_2=1$ and then

$$\Phi_{v_0}(g_2^{-1}g_1,g_1^{-1}g_2)=\int_{H_{v_0}}f_{v_1}(hg_1)f_{v_2}(hg_2)dh.$$

We choose one of the two places, $v_1$ say. Then we may identify $\mathfrak{S}_{v_0}$ with $GL(3,F_{v_0})$, via the map

$$(g_1,g_2)\mapsto g_1.$$

We may also identify $GL(3,E_{v_1})$ and $GL(3,E_{v_2})$ with $GL(3,F_{v_0})$. Then

$$\Phi_{v_0}(g_1)=\int_{GL(3,F_{v_0})}f_{v_1}(hg_1)f_{v_2}(h)dh$$

so that $\Phi_{v_0}$ is the convolution of $f_{v_1}$ and the function $\check{f}_{v_2}$ defined by $\check{f}_{v_2}(g)=f_{v_2}(g^{-1})$. The orbital integral becomes

$$J(zd_a,\Phi_{v_0}):=\int_{N(F_{v_0})\times N(F_{v_0})}\Phi_{v_0}(n_2^{-1}wzd_an_1)\theta(n_2)\theta(n_1)dn_1dn_2, \tag{9}$$

where $z$ is a central element, $d_a=\mathrm{diag}(a_1,1,a_2^{-1})$ and $\theta$ is defined by

$$\theta\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}=\psi_{F_{v_0}}(x-y).$$

Likewise, we consider the unitary group $U$ for the Hermitian matrix $w$. The group is defined by the equation ${}^t\overline{g}wg=w$. We denote by $A'$ the group of diagonal matrices, by $N'$ the group of upper triangular matrices with unit diagonal in $U$ and by $Z_U$ the center of $U$. Let $f'$ be a smooth function of compact support on $U(F_{\mathbb{A}})$. We construct a kernel

$$K'_{f'}(x,y):=\int_{Z_U(F_{\mathbb{A}})/Z_U(F)}\sum_{\xi\in U(F)}f'(x^{-1}\xi yz)\zeta(z)dz, \tag{10}$$

and the distribution

$$J'(f') := \int K_{f'}(n_1, n_2)\theta'(n_1)\theta'(n_2)dn_1 dn_2. \tag{11}$$

Here $\theta'$ is the character of $N'(F_{\mathbb{A}})$ trivial on $N'(F)$ defined by

$$\theta'\begin{pmatrix} 1 & x & t+\frac{x\overline{x}}{2} \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x).$$

The group $A'(F)$ acts by conjugation on $N'$ and thus on the set of characters $N'(F_{\mathbb{A}})$ trivial on $N'(F)$. The group $A'(F)$ is transitive on the set of generic characters. Thus the choice of $\theta'$ is inessential. We will make the following assumptions. Let $S_0$ and $S$ be as above. Then we assume that for every place $v_0 \in S_0$ inert in $E$, the function $f'_{v_0}$ is supported on the open set $\Omega'_{v_0}$ of matrices of the form $n_2^{-1}wud_an_1$. For $u \in U(F_{\mathbb{A}})$ and $a \in E_{\mathbb{A}}^{\times}$ we consider the global orbital integral

$$J'(ud_a, f') := \int f'(n_1^{-1}wud_an_2)\theta(n_1)\theta(n_2)dn_1 dn_2. \tag{12}$$

Then

$$J'(f') = \sum_{\alpha} \int J'(ud_\alpha, f')\zeta(u)du.$$

As before, we have a notion of local orbital integrals. At a place $v_0$ inert in $E$,

$$J'(ud_a, f'_{v_0}) := \int_{N'(F_{v_0})\times N'(F_{v_0})} f'_{v_0}(n_1^{-1}wud_an_2)\theta'(n_1)\theta'(n_2)dn_1 dn_2. \tag{13}$$

If the place $v_0$ splits into $v_1, v_2$ then $U_{v_0}$ is the group of pairs $(g_1, g_2)$ with $g_2 = w^t g_1^{-1} w$. Identifying as before $U_{v_0}$ to $GL(3, F_{v_0})$ via the map $(g_1, g_2) \mapsto g_1$, we see that the local orbital integral is again given by (9) with $\Phi_{v_0}$ replaced by $f'_{v_0}$.

In this paper we will consider pairs $(f, f')$ with matching orbital integrals, in the sense that

$$J(ud_a, \Phi) = J'(ud_a, f'). \tag{14}$$

More precisely, we will assume that for every place $v_0$ of $F$ split in $E$ we have

$$f'_{v_0} = \Phi_{v_0},$$

where both sides are viewed as functions on $GL(3, F_{v_0})$. For $v_0 \in S_0$ inert in $E$, we shall assume that

$$J(ud_a, \Phi_{v_0}) = J'(ud_a, f'_{v_0}). \tag{15}$$

In view of our simplifying assumption, given $\Phi_{v_0}$ there is $f'_{v_0}$ such that (15) is true. Finally, we assume that for $v_0 \notin S_0$ and inert, the function $f'_{v0}$ is the image of the function $f_{v_0}$ by the base change homomorphism of the Hecke algebra. It is a theorem of [J2] and [Mao2] that relation (15) is then true. Thus relation (14) is true as well and we get

$$J(f) = J'(f'). \tag{16}$$

As usual, we decompose the kernel $K$ and $K'$ with respect to cuspidal data. For cuspidal data $\chi$ for $G$ (resp. $\chi'$ for $U$) we obtain a kernel $K_\chi$ (resp. $K'_{\chi'}$). We define a corresponding distribution $J_\chi(f)$ (resp. $J_{\chi'}(f')$). Let $\chi$ be a cuspidal automorphic representation of $GL(3, E_{\mathbb{A}})$ whose central character is $\omega$. Then $J_\chi(f) = 0$ unless $\chi$ is distinguished by $H$. Conversely, we verify in the third section that, if $\chi$ is distinguished by $H$, then there is $f$ satisfying our simplifying assumptions such that $J_\chi(f) \neq 0$.

This being so, we consider a stable packet of automorphic representations of $U$ with central character $\zeta$. The base change of this packet is a single cuspidal representation $\chi$ of $GL(3, E_{\mathbb{A}})$ with central character $\omega$ where $\omega(z) = \zeta(zz^{-\sigma})$. Now $\chi$ is distinguished. It follows that we have an identity

$$J_\chi(f) = \sum_{\chi'} J_{\chi'}(f), \tag{17}$$

where the sum on the right is over all $\chi'$ such that the unramified representation $\bigotimes_{v\notin S} \chi_v$ is the image of the unramified representation $\bigotimes_{v_0\notin S_0} \chi'_{v_0}$ under the standard base change. For the group $U$, this amounts to saying that the sum is over all the members of the tempered stable packet whose image is $\chi$. From the formula, it follows that there is at least one such $\chi'$ such that $J_{\chi'} \neq 0$. This implies that $\chi'$ is globally generic and proves the theorem.

A natural question is to ask whether the sum on the right of (17) has only one element. We can answer this question affirmatively in the last section, thanks to local results. Taking this result for granted, we have then an identity

$$J_\chi(f) = J_{\chi'}(f'). \tag{18}$$

In addition to the previous notations, we will also introduce the following subgroups and elements of $GL(3)$:

$$P := \left\{ \begin{pmatrix} * & * & * \\ ** & * & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad P_1 = PZ, \tag{19}$$

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{20}$$

The paper is arranged as follows. In section 2 we discuss a few needed complements on the Asai $L$-function. In section 3, we prove that the local relative Bessel distributions do not vanish on a suitable open set. In section 4, for the sake of completeness, we establish elementary results on orbital integrals, which are used here, and have also been used in previous references without proof. In section 5, we provide estimates for the functionals at hand. The analysis of the continuous spectrum of the trace formulas occupies sections 6 to 10. The main theorems are quickly established in section 11.

## 2. Complements on the Asai $L$-function

Let $E/F$ be a quadratic extension of number fields with Galois group $\{1,\sigma\}$. Regard the group $GL(n,E)$ as an algebraic group over $F$ via restriction of scalars. The (Galois form of) the $L$-group of $G$ is the semi-direct product:

$$^LG := (GL(n,\mathbb{C}) \times GL(n,\mathbb{C})) \times \{1,\sigma\}.$$

The Asai $L$-function is the Langlands $L$-function attached to the $n^2$ dimensional twisted representation $r$: $^LG \to GL(\mathbb{C}^n \otimes \mathbb{C}^n)$ defined by

$$r(g_1,g_2)(v\otimes w) = (g_1 v)\otimes(g_2 w),$$
$$r(\sigma)(v\otimes w) = w\otimes v.$$

Let $\Pi$ be a cuspidal automorphic representation of $GL(3)$. We fix $S_0$ as before so that $\Pi$ is unramified outside $S$. We denote by $L^{S_0}(s,\Pi,\mathrm{Asai})$ the product of the local factors of the Asai $L$-function over the places **not** in $S_0$.

LEMMA 1: *Let $\mu$ be any idele class character of $E$ such that $\mu|F_{\mathbb{A}}^\times = \omega_{E/F}$. Then, if $\mu$ is unramified outside $S$,*

$$L^S(s,\Pi\otimes\Pi^\sigma) = L^{S_0}(s,\Pi,\mathrm{Asai})L^{S_0}(s,\Pi\otimes\mu,\mathrm{Asai}).$$

*Proof:* The lemma is formal from the point of view of the $L$-group (cf. [HLR] and [Gol]). For the sake of completeness we provide a proof. We claim that the contributions of each place $v_0$ of $F$ not in $S_0$ to the two sides are the same. If $v_0$ is inert and $\Pi_v$ is the unramified representation associated to $n$ unramified quasi-characters $\chi_i$ of $E$, the contribution of $v_0$ to the left hand side is

$$\prod_i L(s,\chi_i^2)\prod_{i<j} L(s,\chi_i\chi_j)^2,$$

while

$$L(s, \Pi_{v_0}, \text{Asai}) = \prod_i L(s, \chi_i|F) \prod_{i<j} L(s, \chi_i\chi_j).$$

Since $\mu_v^2 = 1$, our assertion follows from the relation

$$L(s, \chi_i^2) = L(s, \chi_i|F)L(s, \chi_i|F\omega_{E/F}).$$

See also [Gol]. If $v_0$ splits into two places $v_1, v_2$ then $q_{v_0} = q_{v_1} = q_{v_2}$. If we write $L(s, \Pi_{v_i}) = \det\left[I - t(\Pi_{v_i})q_{v_i}^{-s}\right]^{-1}$ with $t(\Pi_{v_i}) \in GL(n, \mathbb{C})$, then the contribution of $v_0$ to the left hand side is

$$\det\left[I - t(\Pi_{v_1}) \otimes t(\Pi_{v_2})q_{v_0}^{-s}\right]^{-2}.$$

On the other hand,

$$L(s, \Pi_{v_0}, \text{Asai}) = \det\left[I - t(\Pi_{v_1}) \otimes t(\Pi_{v_2})q_{v_0}^{-s}\right]^{-1},$$
$$L(s, \Pi_{v_0} \otimes \mu_{v_0}, \text{Asai}) = \det\left[I - t(\Pi_{v_1} \otimes \mu_{v_1}) \otimes t(\Pi_{v_2} \otimes \mu_{v_2})q_{v_0}^{-s}\right]^{-1}.$$

Our assertion follows from the fact that $\mu_{v_1}\mu_{v_2} = 1$. ∎

The following result is then true in the context of $GL(n)$ with $n$ odd:

PROPOSITION 1: *Let $\Pi$ be a cuspidal automorphic representation of $GL(n)$ with $n$ odd. Suppose that the central character $\omega$ of $\Pi$ is trivial on $F_{\mathbb{A}}^\times$ (that is, is distinguished). Then $\tilde{\Pi} \simeq \Pi^\sigma$ if and only if $\Pi$ is distinguished by $H$.*

*Proof:* We first remark that the relation $\tilde{\Pi} \simeq \Pi^\sigma$ implies $\omega\omega^\sigma = 1$. This is equivalent to $\omega|F_{\mathbb{A}}^\times = 1$ or $\omega|F_{\mathbb{A}}^\times = \omega_{E/F}$.

For now consider an arbitrary cuspidal representation $\Pi$ of $GL(n, E)$ with central character $\omega$. Let $\omega_H$ be the restriction of $\omega$ to $F_{\mathbb{A}}^\times$. Suppose that $\omega_H = 1$ and $\Pi$ is distinguished. Then ([F2]) $\tilde{\Pi} \simeq \Pi^\sigma$.

Conversely, assume that $\Pi$ is a cuspidal automorphic representation of $GL(n)$. It is a result of [F1] and [FZ] that the partial Asai $L$-function has a pole at $s = 1$ if and only if $\omega_H = 1$ and $\Pi$ is distinguished. Suppose that $\tilde{\Pi} \simeq \Pi^\sigma$ and $\omega_H = 1$. Then the partial Rankin–Selberg $L$-function has a pole at $s = 1$. From the previous lemma it follows that either the partial Asai $L$-function attached to $\Pi$ has a pole at $s = 1$ or the partial Asai $L$-function attached to $\Pi \otimes \mu$ has a pole at $s = 1$. But the restriction of the central character of $\Pi \otimes \mu$ to $F_{\mathbb{A}}^\times$ is $\omega_{E/F} \neq 1$. Thus the Asai $L$-function attached to $\Pi$ has a pole and $\Pi$ is distinguished. ∎

*Remark 1:* By the results of Shahidi on $L$-functions attached to Eisenstein series, the Asai $L$-functions do not vanish at $s = 1$. We could use this to give a global proof (rather than a local proof as in [F2]) of the only if part of the proposition.

*Remark 2:* Likewise, for $n$ even, then $\Pi$ satisfies $\tilde{\Pi} = \Pi^\sigma$ if and only if either $\Pi$ is distinguished or $\Pi \otimes \mu$ is distinguished. In particular, the central character of $\Pi$ is then distinguished; this gives a positive answer to a question raised in [Ro1]. On the other hand, according to the discussion in [F2], in the first case $\Pi$ should be the unstable base change of a stable cuspidal packet; in the second case, it should be the stable base change of a stable cuspidal packet.

If $\Pi$ is distinguished we will need to express the linear form

$$\mathcal{J}(\phi) := \int_{Z_H(F_{\mathbb{A}})H(F)\backslash H(F_{\mathbb{A}})} \phi(h)dh$$

on the space of $\Pi$ as a product of local linear forms. When the real places of $F$ split in $E$, the uniqueness theorem of [F2] implies the existence of such a decomposition. We will however give an explicit decomposition without appealing to the uniqueness result. To that end, we review the construction of [F1]. Let $\Phi$ be a Schwartz–Bruhat function on $F^n_{\mathbb{A}}$ and $\phi$ a vector in $\Pi$. Set

$$W(g) = \int \phi(ng)\overline{\theta}(n)dn.$$

We set

$$\Psi(s, W, \Phi) := \int_{N(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} W(\epsilon h)\phi[(0, 0, \dots, 1)h] \mid \det h \mid^s dh.$$

The rational diagonal matrix $\epsilon = \mathrm{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in A(E)$ is so chosen that $\mathrm{tr}(\epsilon_i/\epsilon_{i+1}) = 0$, for every $i$. The integrand is indeed invariant under $N(F_{\mathbb{A}})$. We assume that $W$ and $\Phi$ are product of local functions. We choose $S_0$ sufficiently large so that outside $S$ the function $W_v$ is $K_v$-invariant (equal to 1 on $K_v$) and outside $S_0$, $\Phi_{v_0}$ is the characteristic function of $\mathcal{O}^n_{v_0}$. Moreover, $\epsilon$ is in $K_v$ for $v \notin S$. If $v_0$ is inert in $E$ and $v$ is the corresponding place of $E$, we introduce a local integral, which is absolutely convergent for $\Re s \geq 1$, and, in fact, in a half strip $\Re s > 1 - \epsilon$ ([F2], [FZ]):

$$\Psi(s, W_v, \Phi_{v_0}) := \int_{N(F_{v_0})\backslash H(F_{v_0})} W_v(\epsilon h)\Phi_{v_0}[(0, \dots, 0, 1)h] \mid \det h \mid^s dh.$$

If $v_0$ splits into two places $v_1, v_2$ then we introduce a local integral which converges for $\Re s \geq 1$ and, in fact, in a half strip $\Re s > 1 - \epsilon$ ([JS]),

$$\Psi(s, W_{v_1}, W_{v_2}, \Phi_{v_0}) :=$$

$$\int_{N(F_{v_0})\backslash H(F_{v_0})} W_{v_1}(\epsilon_{v_1}h)W_{v_2}(\epsilon_{v_2}h)\Phi_{v_0}[(0,\ldots,0,1)h] \mid \det h \mid^s dh.$$

Let $S_i$ be the set of places in $S_0$ which are inert in $E$. Likewise, let $S_s$ be the set of places in $S_0$ which split in $E$. Then ([F1])

$$\int \phi(h)\left(\sum_{\xi\in F^n-0}\Phi(\xi h)\right) \mid \det h \mid_F^s dh = \Psi(s,W,\Phi)$$
$$= L^{S_0}(s,\Pi,\mathrm{Asai}) \times \prod_{v_0\in S_i}\Psi(s,W_v,\Phi_{v_0}) \prod_{v_0\in S_s}\Psi(s,W_{v_1},W_{v_2},\Phi_{v_0}).$$

Taking the residue at $s=1$, we get

$$c\mathcal{J}(\phi)\int\Phi(x)dx = \mathrm{Res}_{s=1}L^{S_0}(s,\Pi,\mathrm{Asai}) \times \prod_{v_0\in S_i}\Psi(1,W_v,\Phi_{v_0}) \prod_{v_0\in S_s}\Psi(1,W_{v_1},W_{v_2},\Phi_{v_0}),$$

where the constant $c>0$ depends only on the choice of the Haar measures. For $v_0\in S_i$, let us set

$$\mathcal{J}_{v_0}(W_v) := \int_{N(n-1,F_{v_0})\backslash GL(n-1,F_{v_0})} W_v(\epsilon h)dh.$$

Here we embed $GL(n-1)$ into $GL(n)$ as a factor of the Levi-subgroup of type $(n-1,1)$. The integral is absolutely convergent ([F2], [FZ]). It is clear that the linear form $\mathcal{J}_{v_0}$ is invariant under $P_1(F_{v_0})$. We claim that it is actually invariant under $GL(3,F_{v_0})$. Indeed

$$\Psi(1,W_v,\Phi_{v_0}) = \int_{K_{v_0}}\int_{F_{v_0}^\times}\mathcal{J}_{v_0}(\rho(k)W_v)\Phi_{v_0}[(0,0,\ldots,t)k] \mid t\mid^n d^\times t dk.$$

On the other hand, for a suitable choice of the Haar measures,

$$\int\Phi_{v_0}(x)dx = \int_{K_{v_0}}\int_{F_{v_0}^\times}\Phi_{v_0}[(0,0,\ldots,t)k] \mid t\mid^n d^\times t dk.$$

It follows that there is a linear form $\gamma$ on the Whittaker model of $\pi_v$ such that

$$\int_{K_{v_0}}\int_{F_{v_0}^\times}\mathcal{J}_{v_0}(\rho(k)W_v)\Phi_{v_0}[(0,0,\ldots,t)k] \mid t\mid^n d^\times t dk$$
$$= \gamma(W_v)\int_{K_{v_0}}\int_{F_{v_0}^\times}\Phi_{v_0}[(0,0,\cdots,t)k] \mid t\mid^n d^\times t dk,$$

for any $W_v$ and any function $\Phi_{v_0}$. This implies that for any smooth function $f$ on $K_{v_0}$ which is invariant under $K_{v_0} \cap P_{1,v_0}$ on the left

$$\int_{K_{v_0}} \int_{F_{v_0}^\times} \mathcal{J}_{v_0}(\rho(k)W_v)f(k)dk = \gamma(W_v) \int_{K_{v_0}} f(k)dk.$$

The same relation is then true for any smooth function $f$ on $K_{v_0}$. It follows that $\gamma = \mathcal{J}_{v_0}$ and that $\mathcal{J}_{v_0}$ is invariant under $K_{v_0}$. Thus it is in fact invariant under $GL(n, F_{v_0})$. Likewise, if $v_0$ splits into $v_1, v_2$ then an invariant linear form on $\Pi_{v_1} \otimes \Pi_{v_2}$ is given by

$$\mathcal{J}_{v_0}(W_{v_1} \otimes W_{v_2}) := \int_{N(n-1,F_{v_0})\backslash GL(n-1,F_{v_0})} W_{v_1}(\epsilon_1 h)W_{v_2}(\epsilon_2 h)dh.$$

In fact this is known (see [B], [Ba], and the next section). For a suitable positive constant $c'$, we get

$$c'\mathcal{J}(\phi) = \mathrm{Res}_{s=1} L^{S_0}(s, \Pi, \mathrm{Asai})) \prod_{v_0 \in S_i} \mathcal{J}_{v_0}(W_v) \prod_{v_0 \in S_s} \mathcal{J}_{v_0}(W_{v_1} \otimes W_{v_2}).$$

Note that the global linear form $\mathcal{J}$ is continuous for the topology of the smooth vectors on the space of the representation $\Pi$. In a precise way, let $\mathcal{H}$ be the Hilbert space of square integrable elements of $\Pi$ and $\mathcal{V}$ be the space of smooth vectors in $\mathcal{H}$. Let $K'$ be a compact open subgroup of $G^\infty$, then the representation of $G_\infty$ on $\mathcal{H}^{K'}$ is a finite multiple of $\Pi_\infty$ and $\mathcal{V}^{K'}$ is the space of smooth vectors of this representation. The topology of $\mathcal{V}^{H'}$ is the topology defined by the semi-norms $\phi \mapsto |\ \rho(X)\phi|\ _2$. The space $\mathcal{V}$ is the inductive limit of the spaces $\mathcal{V}^{K'}$. The linear form $\mathcal{I}$ is continuous on each one of the spaces $\mathcal{V}^{K'}$, or, what amounts to the same, continuous on $\mathcal{V}$. Likewise for an infinite place $v_0 \in S_i$ which is below a place $v$ of $E$, the linear form $\mathcal{I}_{v_0}$ may be viewed as a continuous linear form on the space of smooth vectors of $\Pi_v$. For an infinite place $v_0 \in S_s$ which split into $v_1, v_2$, the linear form $\mathcal{I}_{v_0}$ may be viewed as a continuous linear form on the space of smooth vectors of $\Pi_{v_1} \otimes \Pi_{v_2}$. Indeed this follows at any rate from the above formula. We define the **relative global Bessel distribution** $\mathcal{J}_\Pi$ as follows: if $f$ is a smooth function of compact support, $K$-finite on both sides, then we set

$$\mathcal{J}_\Pi(f) = \sum_i \mathcal{J}(\pi(f)v_i)\overline{\mathcal{W}(v_i)},$$

where $v_i$ is an orthonormal basis of $\mathcal{H}$ whose elements are $K$-finite vectors in $\mathcal{V}$. More precisely, let $\mathcal{V}^*$ be the topological dual vector space of $\mathcal{V}$. We have thus continuous inclusions with dense image

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*.$$

The restriction of the pairing $\langle \bullet, \bullet \rangle$ of $\mathcal{H} \times \mathcal{H}$ to $\mathcal{V} \times \mathcal{V}$ extends to a sesquilinear form denoted in the same way on the product $\mathcal{V} \times \mathcal{V}^*$ or the product $\mathcal{V}^* \times \mathcal{V}$. With this notation we may write $\mathcal{W}(v) = \langle v, \mathcal{W} \rangle$ and then

$$\langle \Pi(n)v, \mathcal{W} \rangle = \theta(n)\langle v, \mathcal{W} \rangle, \langle v, \mathcal{W} \rangle = \overline{\langle \mathcal{W}, v \rangle},$$

and the above formula reads

$$\mathcal{J}_\Pi(f) = \sum_i \langle \pi(f)v_i, \mathcal{J} \rangle \langle \mathcal{W}, v_i \rangle.$$

We still denote by $\pi$ the representation of $G(E_{\mathbb{A}})$ on $\mathcal{V}^*$. If $f$ is a smooth function of compact support on $G(E_{\mathbb{A}})$ then $\Pi(f)(\mathcal{V}^*) \subset \mathcal{V}$. If $f$ is $K$-finite on both sides, we have

$$\mathcal{J}_\Pi(f) = \langle \mathcal{W}, \pi(f^*)\mathcal{J} \rangle = \langle \pi(f)\mathcal{W}, \mathcal{J} \rangle.$$

For arbitrary $f$ we take this to be the definition of $\mathcal{J}_\Pi(f)$.

For $v_0 \in S_i$, we introduce similarly a local **relative local Bessel distribution** $\mathcal{B}_{v_0}$. For $f_{v_0}$ $K_{v_0}$-finite it is given by

$$\mathcal{B}_{v_0}(f) = \sum_\phi \mathcal{J}_{v_0}(\Pi_{v_0}(f)\phi)\overline{\mathcal{W}_{v_0}(\phi)}$$

where the sum is over an orthonormal basis. There is a similarly defined distribution at the places in $S_s$. Assume $f = f_S f^S$ where $f^S$ is the characteristic function of $K^S$. We find then that the global distribution $J_\Pi(f)$ can be written

$$J_\Pi(f) = C(\Pi) \prod_{v_0 \in S_0} \mathcal{B}_{v_0}(f_{v_0}),$$

where the constant $C(\Pi)$ can be computed explicitly. Unfortunately, we have not established the identity (18) for all functions $f$. Thus we cannot use the previous decomposition to obtain the corresponding decomposition of the right hand side of (18), as we would like to.

## 3. Local relative Bessel distributions

Now we let $E/F$ be a local quadratic extension. We set $P_1 = P(E)Z(E)$ and $H_1 = H(F)Z(F)$. The elements $\{w, w_1, w_2, e\}$ form a set of representatives for the orbits of $B(E)$ on $\mathfrak{S}(F)$. Let $\Omega_w$ be the set of $s$ such that $s_{1,3} \neq 0$, $\Omega_1$ (resp. $\Omega_2$) the set of $s \in \mathfrak{S}(F)$ such that $s_{1,2} \neq 0$ or $s_{1,3} \neq 0$ (resp. $s_{1,3} \neq 0$ or $s_{2,3} \neq 0$). These sets are open. For $s \in \mathfrak{S}(F)$ we have

$$\bar{s}_{1,3} = \frac{D_1}{\det s}, \quad D_1 = \begin{vmatrix} s_{1,2} & s_{2,2} \\ s_{1,3} & s_{2,3} \end{vmatrix}.$$

Thus for $s \in \Omega_w$ we have also $D_1 \neq 0$. It follows that $s$ can be written in the form $s = b_1 w b_2$ with $b_i \in B(E)$. In turn this implies that $s = bw\bar{b}^{-1}$ with $b \in B(E)$. Thus $\Omega_w$ is the orbit of $w$ under $B(E)$. We note that for $s \in \mathfrak{S}(F)$ the relations $s_{1,3} = 0$ and $s_{1,2} \neq 0$ imply $D_1 = 0$ and then $s_{2,3} = 0$. It easily follows that $O_1 := \Omega_1 - \Omega_w = P_1(E) \cap \mathfrak{S}(F)$ is the orbit of $w_1$ under $B(E)$. Likewise $O_2 := \Omega_2 - \Omega_w$ is the orbit of $w_2$ under $B(E)$. Finally $\mathfrak{S}(F)$ is the union of the open sets $\Omega_*$ and the closed set $O_e := B(E) \cap \mathfrak{S}(F)$, which is in fact the orbit of $e$ under $B(E)$. The orbit of $w$ under $P_1$ contains $w$ and $w_1 w w_1 = w_2$ but not $w_1$. It follows that $\Omega_2$ is the orbit of $w$ under $P_1$.

We consider a unitary irreducible representation $\pi$ of $GL(3)$ on a Hilbert space $\mathcal{H}$. We denote by $\mathcal{V}$ the space of smooth vectors in $\mathcal{H}$. We assume that $\pi$ is distinguished by $H(F)$ in the sense that there is a non-zero continuous linear form $\mathcal{J}$ on $\mathcal{V}$ such that $\mathcal{J}(\pi(h)v) = \mathcal{J}(v)$ for all $v \in \mathcal{V}$ and all $h \in H(F)$. We denote by $\omega$ the central character of $\pi$; it is necessarily trivial on $F^\times$, that is, distinguished. We denote in the same way the extension of $\omega$ to $H_1$ which is trivial on $H(F)$. Here, it is more convenient to let $\psi$ be a non-trivial character of $E$ trivial on $F$ and define a character $\theta$ of $N(E)$ by $\theta(n) = \psi(n_{2,1} + n_{2,3})$. We assume that $\pi$ is generic and let $\mathcal{W}$ be a continuous non-zero linear form on $\mathcal{V}$ such that $\mathcal{W}(\pi(n)v) = \theta(n)\mathcal{W}(v)$, for all $n \in N(E)$ and all vectors $v$. We define the corresponding **relative Bessel distribution** $\mathcal{B}$ as follows: if $f$ is a smooth function of compact support, $K$-finite on both sides, then

$$\mathcal{B}(f) = \sum_i \mathcal{J}(\pi(f)v_i)\overline{\mathcal{W}(v_i)},$$

where $v_i$ is an orthonormal basis of $\mathcal{H}$ whose elements are **K**-finite vectors in $\mathcal{V}$. As before, we let $\mathcal{V}^*$ be the topological dual vector space of $\mathcal{V}$. We have thus continuous inclusions with dense image:

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*.$$

The restriction of the pairing $\langle \bullet, \bullet \rangle$ of $\mathcal{H} \times \mathcal{H}$ to $\mathcal{V} \times \mathcal{V}$ extends to a sesquilinear form denoted in the same way on the product $\mathcal{V} \times \mathcal{V}^*$ or the product $\mathcal{V}^* \times \mathcal{V}$. We still denote by $\pi$ the representation of $G(E)$ on $\mathcal{V}^*$. If $f$ is a smooth function of compact support on $G(E)$ then $\pi(f)(\mathcal{V}^*) \subset \mathcal{V}$. If $f$ is $K$-finite on both sides, we have

$$\mathcal{B}(f) = \langle \mathcal{W}, \pi(f^*)\mathcal{J} \rangle = \langle \pi(f)\mathcal{W}, \mathcal{J} \rangle.$$

For arbitrary $f$ we take this for definition of $\mathcal{B}(f)$. Here and below we use an integral notation for the value of $\mathcal{B}$ on a function $f$. The distribution $\mathcal{B}$ has the following properties of invariance: for $h \in H_1$ and $n \in N(E)$,

$$\int f(hxn)d\mathcal{B}(x) = \theta(n)^{-1}\omega(h)^{-1}\int f(x)d\mathcal{B}(x).$$

In this section, we prove the following result:

THEOREM 1: *The restriction of $\mathcal{B}$ to the open set $\Omega_F := \mathcal{P}^{-1}(\Omega_w)$ is non-zero.*

3.1 PROOF: FIRST STEP. We prove the theorem when $E/F$ is the extension $\mathbb{C}/\mathbb{R}$. The proof in the non-Archimedean case is similar but simpler. We assume that the restriction of $\mathcal{B}$ to $\Omega_F$ is zero and prove that $\mathcal{B}$ is then zero. This will give a contradiction and prove the theorem. Since $\mathcal{B}$ is invariant on the left under $H(F)$ it determines a distribution $\Xi$ on $\mathfrak{S}(F)$: if $\Phi$ is a smooth function of compact support on $\mathfrak{S}(F)$ then $\Xi(\Phi) = \mathcal{B}(f)$ where $f$ is any smooth function of compact support on $G(E)$ such that

$$\Phi(\mathcal{P}(g)) = \int_{H(F)} f(hg)dh.$$

Thus $\Xi$ vanishes on the set $\Omega_w$. We first show that it vanishes on $\Omega_1$ and $\Omega_2$.

To that end, we recall, in the form appropriate to us, some results of [KV]. Let $H$ be a Lie group operating on the right on a manifold $X$ and let $H'$ be a closed normal subgroup of $H$. We denote by $\sigma$ the action of $H$ on $X$. We also denote by $\sigma$ the corresponding left action of $G$ on functions on $X$:

$$\sigma(h)\Phi(x) = \Phi(x\sigma(h)).$$

In our case $H' = Z(\mathbb{C})N(\mathbb{C})$ and $H = B(\mathbb{C})$. In particular, for our purposes, we may assume that $H'$ and all of its closed subgroups are unimodular. Let $\theta$ be a one dimensional character of $H'$. Suppose that $O$ is a closed orbit of $H$ in $X$. Let $T$ be a distribution on $x$, transforming according to $\theta$ under $H'$ and supported on $O$. Thus

$$T(\sigma(h')\Phi) = \theta(h')^{-1}T(\Phi).$$

Let $M^{(r)}(O)$ or simply $M^{(r)}$ be the $r$-th graded component of the transverse jet bundle of $O$. Thus $M^{(1)}$ is the normal tangent bundle and $M^{(r)}$ is its $r$-th symmetric power. Then ([KV]) the support of $T$ is contained in the set of $y \in O$ such that for some $r$ the fiber $M_y^{(r)}$ contains a non-zero vector transforming under the character $\theta$ restricted to $H'_y$, the stabilizer of $y$ in $H'$.

Going back to the proof of our theorem, we assume that the distribution $\Xi$ vanishes on the open set $\Omega_w$. We apply the just recalled results to the group

$H = B(\mathbb{C})$ and the subgroup $H' = Z(\mathbb{C})N(\mathbb{C})$. If $y$ is any point of $\mathfrak{S}(F)$ the stabilizer of $y$ in $H'$ is the product of $Z(\mathbb{R})$ and the stabilizer $N_y$ of $y$ in $N(\mathbb{C})$. As a matter of fact $Z(\mathbb{R})$ operates trivially on $\mathfrak{S}(F)$. We first show that the restriction of $\Xi$ to $\Omega_1$ vanishes. Since $\Omega_1 = O_1 \cup \Omega_w$ and $O_1$ is closed in $\Omega_1$ this will follow from the results we recalled above and the following lemma:

LEMMA 2: *If $y \in O_1$ then the restriction of $\theta$ to $N_y$ is non-trivial and the action of $N_y$ on $M_y^{(r)}$ is unipotent.*

*Proof:* Since $B(\mathbb{C})$ is transitive on $O_1$ it will suffice to prove the two assertions for the point $y = w_1$. Then $N_y$ is the group of matrices of the form

$$n = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \mathbb{C}.$$

The first assertion follows at once. For the second assertion, we remark that the action of $N_y$ on $M_y^1$ is algebraic, hence unipotent. Thus the same is true of the action on $M_y^{(r)}$. (In fact, one can check that the action on $M_y^1$ is trivial.) ∎

The lemma being established it follows that the restriction of $\Xi$ to $\Omega_1$ is trivial. Likewise the restriction to $\Omega_2$ is trivial. Equivalently, the restriction of $\mathcal{B}$ to $\Omega = \mathcal{P}^{-1}(\Omega_2)$ is trivial.

3.2 PROOF: SECOND STEP. We first recall some results on generic representations of $GL(3)$. We denote by $\mathcal{W}(\pi)$ the space spanned by the functions $g \mapsto \mathcal{W}(\pi(g)v)$, with $v \in \mathcal{V}$. It is convenient to identify $\mathcal{V}$ with $\mathcal{W}(\pi)$.

LEMMA 3: *For every $W \in \mathcal{W}(\pi)$ the restriction of $W$ to $P(E)$ is square integrable modulo $N(E)$. Moreover, the invariant (positive definite) scalar product on $\mathcal{W}(\pi)$ may be taken to be*

$$\langle W_1, W_2 \rangle = \int_{N(E)\backslash P(E)} W_1 \overline{W}_2(p) d_r p.$$

*In particular, the map $W \mapsto W|P$ is injective.*

*Proof:* In the non-Archimedean case this is a result of Bernstein valid for $GL(n)$ ([B]). In the Archimedean case, the result follows from the global theory if the representation is a component of a cusp form: the proof is similar to the proof given in the previous section for the invariance of $\mathcal{I}$. At any rate, the result also follows from [JS] and a recent work of Baruch ([Ba]). ∎

LEMMA 4: *Let $\mathcal{U}$ or $\mathcal{U}(\pi)$ be the space of $W \in \mathcal{W}(\pi)$ such that the function $W|P(E)$ has compact support modulo $N(E)$. Then the space $\mathcal{U}$ is non-zero.*

*Proof:* In the non-Archimedean case, this is a result of Gelfand and Kazhdan ([GK]). The proof in the Archimedean case is similar once we observe the following: let $V$ be a commutative unipotent subgroup of $GL(3,E)$, so that $V$ is isomorphic to a real vector space as a real Lie group; then for every Schwartz function $\phi$ on $V$ and every smooth vector $u$ in the space of $\pi$ the vector

$$\pi(\phi)u := \int_V \phi(v)\pi(v)udv$$

is still a smooth vector. ∎

By definition

$$\langle W, \pi(f^*)\mathcal{W}\rangle = \int W(g)f(g)dg.$$

Let $W_0$ be an element of $\mathcal{U}$ and and let $\phi_0$ be a smooth function of compact support on $P(E)$ such that

$$\int \phi_0(np)\overline{\theta}(n)dn = W_0(p).$$

Let also $\phi_1$ be a smooth function of compact support on $H_1$ such that $\int_{H_1}\phi_1(h)\omega(h)dh \neq 0$. Since the map $(p,h_1) \mapsto p\eta^{-1}h_1^{-1}$ is submersive with image $\Omega_F^{-1}$, there is a smooth function of compact support $f$, supported on $\Omega_F$, such that for any continuous function $\Psi$ on $G(E)$,

$$\int \Psi(g)f(g^{-1})dg = \int \Psi(p\eta^{-1}h_1^{-1})\phi_0(p)\phi_1(h_1)d_rpdh_1.$$

In particular, for any $W \in \mathcal{W}(\pi)$,

$$\langle \Pi(f)\mathcal{W}, W\rangle = \int \overline{W}(g)f(g^{-1})dg$$

$$= \int_{H_1}\int_P \overline{W}(p\eta^{-1}h^{-1})\phi_0(p)d_rp\phi_1(h)dh = \int \langle W_0, \Pi(\eta^{-1}h^{-1})W\rangle\phi_1(h_1)dh_1$$

$$= \langle \int \Pi(h)\phi(h)dh\Pi(\eta)W_0, W\rangle.$$

Equivalently

$$\Pi(f)\mathcal{W} = \int \Pi(h)\phi_1(h)dh\Pi(\eta)W_0,$$

and thus

$$\mathcal{B}(f) = \int \omega(h)\phi_1(h)dh\langle W_0, \mathcal{J}\rangle.$$

Now $\mathcal{B}(f) = 0$. By our choice of $\phi_1$ we find

$$\langle \Pi(\eta) W_0, \mathcal{J} \rangle = 0.$$

This relation takes place for all $W_0 \in \mathcal{U}$. Since $\mathcal{U}$ is invariant under $P(E)$ we have also

$$\langle \pi(g) W_0, \mathcal{J} \rangle = 0$$

for all $g \in \Omega_F$. Since $\Omega_F$ is dense and the representation $\Pi$ on $\mathcal{V}$ is topologically irreducible, we conclude that $\mathcal{J} = 0$, a contradiction. ∎

3.3 A RESULT ON LOCAL DISTINGUISHED REPRESENTATIONS. The following proposition was part of the thesis of Youngbin Ok, written under the direction of H. Jacquet. It is not really needed in this paper, since, as noted, the result can be established by global means for the components of a distinguished cuspidal representation. Nonetheless, it is interesting that it can be established by purely local means in the context of $GL(n, E)$, where $E/F$ is a quadratic extension of local non-Archimedean fields. It is analogous to the main result of [B].

PROPOSITION 2: *Let $\pi$ be a generic unitary irreducible representation of $GL(n, E)$, distinguished by $GL(n, F)$. We identify the space $\mathcal{V}$ of smooth vectors to the Whittaker model of $\pi$ and denote by $\mathcal{J}$ a non-zero continuous linear form on $\mathcal{V}$ invariant under $GL(n, F)$. We assume that the generic character $\theta$ is trivial on $N(F)$. Let $P$ be the subgroup of matrices with last row $(0, 0, \ldots, 0, 1)$ and $N$ the group of upper triangular matrices with unit diagonal. Then there is a constant $c \neq 0$ such that*

$$\mathcal{J}(W) = c\lambda(W),$$

*where*

$$\lambda(W) := \int_{N(F) \backslash P(F)} W(p) dp,$$

*and $dp$ is a right invariant measure on $N(F) \backslash P(F)$.*

*Proof:* Indeed, according to [B], page 82, the central exponents of the derivatives $\pi^{[k]}$ of $\pi$, $1 \leq k < n$, are strictly positive. This implies ([F1]) that $\lambda$ is defined by a convergent integral. Clearly, the linear form $\lambda$ is invariant under $P_1(F) = P(F)Z(F)$. It will suffice to show that it is invariant under $G(F)$. It amounts to the same to show that the distribution $\Xi$ defined by

$$\Xi(f) = \sum_i \mathcal{J}(\pi(f) v_i) \overline{\lambda(v_i)}$$

is invariant on the right under $G(F)$. A priori, it is invariant on the left under $G(F)$ and on the right under $P_1(F)$. As before, it defines a distribution $\Xi_{\mathfrak{S}}$ on $\mathfrak{S}(F)$, invariant under $P_1(F)$. The following lemma implies that $\Xi_{\mathfrak{S}}$ and thus $\Xi$ is invariant under $G(F)$.

LEMMA 5: *Any distribution on $\mathfrak{S}(F)$ which is invariant under $P_1(F)$ is also invariant under $G(F)$.*

We sketch a proof of this lemma. By the results of [B], the analogous assertion is true for the tangent space of $\mathfrak{S}(F)$, that is, the space $\mathfrak{S}_0$ of matrices $X \in M(n \times n, E)$ such that $X + \overline{X} = 0$. Let $U \subset \mathfrak{S}_0$ be the open set of $X$ such that $\det(1+X).\det(1-X) \neq 0$. Suppose that $\mu$ is a distribution on $U$ invariant under $P_1(F)$. Then for any smooth function of compact support $f$ on $F^\times$ the product of $\mu$ and the function $f(\det(1+X).\det(1-X))$ is the restriction to $U$ of a distribution $\mu_f$ on $\mathfrak{S}_0$. The distribution $\mu_f$ is supported on $U$ and invariant under $P_1(F)$. Thus $\mu_f$ is invariant under $G(F)$. It follows that $\mu$ is invariant under $G(F)$. This being so, for each $\lambda \in E^\times$ with $\lambda\overline{\lambda} = 1$, let $\Omega_\lambda$ be the open set of $s \in \mathfrak{S}(F)$ such that $\det(s + \lambda 1_n) \neq 0$. The set $\Omega_\lambda$ is invariant under $G(F)$ and the map

$$\iota_\lambda(s) := \frac{s-\lambda}{s+\lambda}$$

is an analytic bijection of $\Omega_\lambda$ onto $U$, which commutes to the action of $G(F)$. Now if $\mu$ is a distribution on $\mathfrak{S}(F)$ invariant under $P_1(F)$, then for each $\lambda$ the distribution $\iota_\lambda^*(\mu|\Omega_\lambda)$ is invariant under $G(F)$. The same is true of $\mu|\Omega_\lambda$. Since the open sets $\Omega_\lambda$ cover $\mathfrak{S}(F)$ our assertion follows. ∎

## 4. Complements on orbital integrals

We first prove in the context of $GL(n)$ an elementary result which was used without proof in [J2] and [Mao2] in the case of $GL(3)$. Assume that $E/F$ is an unramified quadratic extension of non-Archimedean local fields. Suppose that the residual characteristic is not 2. Denote by $\varpi$ a fixed prime element in the ring of integers $\mathcal{O}_F$. Denote by $\overline{E}/\overline{F}$ the residual quadratic extension. Finally, let $G = GL_n(E)$ and, as before, let $\mathfrak{S}$ be the symmetric space of elements $x \in G$ satisfying $x\overline{x} = 1$. Let $K = GL_n(\mathcal{O}_E)$, let $A$ be the diagonal subgroup, and $w = (\delta_{i,n+1-i})$ or $w_n$ be the standard representative of the longest element in the Weyl group $W$.

PROPOSITION 3: *Every $K$ orbit in $\mathfrak{S}$ contains a unique element of the form $aw$ with $a = \mathrm{diag}(\varpi^{m_1}, \ldots, \varpi^{m_n})$, $m_1 \geq m_2 \geq \cdots \geq m_n$.*

*Proof:* Our assertion being trivial for $n = 1$, we may assume $n > 1$ and our assertion true for $n' < n$. By the Cartan decomposition, we may write $x = k_1 a k_2$ where $k_j \in K$ and $a$ is a diagonal matrix with entries $\varpi^{m_1}, \dots, \varpi^{m_n}$. The element $a$ is uniquely determined up to conjugation by an element of the Weyl group. Furthermore, $a = \overline{a}$ and we may assume that

$$m_1 \geq m_2 \geq \cdots \geq m_n. \tag{21}$$

Now $k_1 a k_2 = \overline{k_2}^{-1} a^{-1} \overline{k_1}^{-1}$. The uniqueness of the Cartan decomposition implies that $a^{-1} = u^{-1} a u$ for some $u$ in the Weyl group $W$. Our hypothesis (21) implies that we may take $u = w$. Thus we can write $x = k_1 a w k_2$, with $k_i \in K$, $aw \in \mathfrak{S}$. Our task is then to show that we can choose $k_1, k_2$ so that $k_2 = \overline{k_1}^{-1}$.

Suppose first that $a = 1$, that is, $x \in K$. Our assertion amounts to saying that $x = k\overline{k}^{-1}$ with $k \in K$. Let $x'$ be the class of $x$ modulo $\varpi \mathcal{O}_E$. Then $x' = g_1 . \overline{g_1}^{-1}$, $g_1 \in GL(n, \overline{E})$. If $k_1$ is congruent to $g_1$ we thus have $x = k_1 s_1 \overline{k_1}^{-1}$ where $s_1 \equiv 1_n$ (mod $\varpi \mathcal{O}_E$). Thus $k_2 = s_1 + 1$ is in $K$ and $k_2 \overline{k_2}^{-1} = s_1$. This proves our assertion in this case.

We pass to the general case where $a \neq 1$, that is, $m_1 > 0$. We let $r \geq 1$ be the integer such that

$$m_1 = m_2 = \cdots = m_r > m_{r-1}.$$

Since we are only interested in $K$-orbits, we may assume that $k_2 = 1$. Then $a^{-1} k_1 a = w \overline{k_1}^{-1} w$ is in $K$. Let $Q$ be the standard parabolic subgroup of type $(r, n-2r, r)$ if $n > 2r$ and type $(r, r)$ if $n = 2r$. Denote by $M$ its standard Levi-factor, $U$ its unipotent radical and $V$ the unipotent radical of the opposite parabolic subgroup. The above relation implies that $k_1$ is congruent to an element of $MV$. Thus we can write $k_1 = v_1 m u_1$ with $v_1 \in V \cap K$, $m \in M \cap K$ and $u_1 \in U \cap K$. We then have $x = v_1 m a w v_2$ with $v_2 \in V$. The uniqueness of this decomposition implies that $v_2 = \overline{v_1}^{-1}$. We are thus reduced to the case where $x = maw$. If $n > 2r$ then

$$x = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & w_r \\ 0 & w_{n-2r} & 0 \\ w_r & 0 & 0 \end{pmatrix}$$

with $m_1, m_3 \in GL(r, \mathcal{O}_E)$, $m_2 \in GL(n-2r, \mathcal{O}_E)$. Set

$$k = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $x$ is equal to

$$k\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & \overline{a}_1^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 & w_r \\ 0 & w_{n-2r} & 0 \\ w_r & 0 & 0 \end{pmatrix}\overline{k}^{-1}.$$

The conclusion follows from the induction hypothesis applied to the element

$$m_2 a_2 w_{n-2r}$$

of the symmetric sapce for $GL(n-2r)$. The case $n = 2r$ is similar. ∎

To prove the convergence of the local and global orbital integrals, as well as the convergence of the geometric part of our relative trace formulas we use the following lemma, that we prove in the context of $GL(n)$:

LEMMA 6: *Let $E$ be a local field and $\Omega$ be a compact set of $GL(n, E)$. Let $A$ be the group of diagonal matrices, $N$ the group of upper triangular matrices with unit diagonal. The relations $n_i \in N(E)$, $a \in A(E)$,*

$$^t n_1 a n_2 \in \Omega$$

*imply that $\det a$ is in a compact set of $E^\times$, the entries of $a$ in a compact set of $E$, and $n_1, n_2$ in compact sets of $N(E)$. If $E$ is non-Archimedean, let $K = GL(n, \mathcal{O}_E)$. Then the relations $a \in A(E)$, $n_i \in N(E)$,*

$$^t n_1 a n_2 \in K, \quad a \in A \cap K$$

*imply that $n_1$ and $n_2$ are in $K$.*

*Proof:* We prove the second assertion. The proof of the first assertion is similar. We recall that the leading principal minors of a matrix are invariant of the action of $N(E) \times N(E)$ on $GL(n, E)$. Thus under the assumptions of the Lemma the leading principal minors of the matrix are units. Our lemma amounts to say that if $x$ is a matrix with integral entries and all its leading principal minors are units (including the determinant) then $x = {}^t n_1 a n_2$ with $n_i \in N \cap K$ and $a \in A \cap K$. Inductively, it suffices to prove that such an $x$ can be written in the form

$$x = \begin{pmatrix} 1_{n-1} & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix}$$

with $g \in GL(n-1, \mathcal{O}_E)$ and $u, v$ integers. Indeed, the right hand side can be written in the form

$$\begin{pmatrix} g & gv \\ ug & * \end{pmatrix}.$$

Thus, at any rate, $x$ can be written in this form and $\det g$ is a unit. In addition $g$ is integral thus $g$ is indeed in $GL(n-1,\mathcal{O}_E)$. Finally $gv$ and $ug$ are integers. Thus the same is true of $u$ and $v$ and our assertion follows. ∎

The lemma can be applied to expressions of the form $n_1wan_2$ and easily implies the desired convergence of the orbital integrals, local and global. The global integrals are product of local integrals, all absolutely convergent and almost all equal to one. In addition, for a given function, only finitely many of the global integrals are non-zero.

## 5. Estimates for functionals

To prove that the relative trace formula is absolutely convergent, it will be convenient to use estimates for the global Whittaker linear forms and period integrals discussed in the introduction. The estimates in question are easily obtained in the case of $GL(2)$ and $GL(3)$. In the context of $GL(n)$ it would not be possible to obtain these estimates and one would have to use more sophisticated methods.

5.1 Integral representation of a section. In this subsection and the two next ones, we discuss the case of the Whittaker linear form and the period integral for $GL(2)$. Thus we let $G$ be the group $GL(2)$. To that end we consider a pair of idele class characters $\sigma=(\sigma_1,\sigma_2)$ of $E$. As usual, the characters are assumed to be trivial on the subgroup $\mathbb{R}_+^\times$ of $E_{\mathbb{A}}^\times$. We let $I(\sigma)$ be the space of smooth functions $\phi\colon G(E_{\mathbb{A}})\to\mathbb{C}$ such that

$$\phi\left[\begin{pmatrix} a_1 & x \\ 0 & a_2\end{pmatrix} g\right]=\sigma_1(a_1)\sigma_2(a_2)\phi(g).$$

We let $K$ be the standard maximal compact subgroup of $G(E_{\mathbb{A}})$ and set

$$|\phi|^2=\int_K |\phi(k)|^2\,dk.$$

For a given $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2$ we set

$$e^{\langle\lambda+\rho,H(g)\rangle}=|a_1|^{1/2+\lambda_1}|a_2|^{-1/2+\lambda_2}$$

if

$$g=\begin{pmatrix} a_1 & x \\ 0 & a_2\end{pmatrix}k,\quad k\in K.$$

We have a fiber bundle of representations $I_{\lambda,\sigma}$. As usual, we identify the space of the representation $I_{\lambda,\sigma}$ to $I(\sigma)$. Then for $\phi\in I(\sigma)$, functions of the form

$$\phi(g;\lambda)=\phi(g)e^{\langle\lambda+\rho,H(g)\rangle}$$

are the **standard sections** of the fiber bundle. We let $f$ be a smooth function of compact support on $GL(2, E_{\mathbb{A}})$ and consider the **convolution section** $I_{\lambda,\sigma}(f)\phi(g;\lambda)$ defined by

$$(I_{\lambda,\sigma}(f)\phi)(g)e^{\langle\lambda+\rho,H(g)\rangle} = \int \phi(gx)e^{\langle\lambda+\rho,H(gx)\rangle}f(x)dx. \tag{22}$$

We first give an integral representation for (22). This is a variation on old ideas which is also described in [CJ]. For the convenience of the reader we go again through the construction. We assume that $f = f_S f^S$ where $f^S$ is the characteristic function of $K^S$. Then if we set

$$\phi_1(g) = \int_{K^S} \phi(gk)dk$$

we see that $\| \phi_1 \| \leq \| \phi \|$ and $I_{\lambda,\sigma}(f)\phi = I_{\lambda,\sigma}(f_S)\phi_1$. The value of our section on $g \in G_S$ is thus given by

$$\int_{G_S} \phi_1(gx)e^{\langle\lambda+\rho,H(gx)\rangle}f_S(x)dx.$$

We set (with $\check{f}(g) = f(g^{-1})$)

$$f_1(g) := \int_{K_S} \phi_1(k)\check{f}(k^{-1}g)dk.$$

Thus $f_1$ is a smooth function of compact support and (22) is equal to

$$\int_{E_S\times E_S^\times} f_1\left[n\begin{pmatrix} a_1 & 0 \\ 0 & a_2\end{pmatrix}g\right] \times | a_1 |^{-1/2-\lambda_1} \sigma_1(a_1)^{-1} | a_2 |^{1/2-\lambda_2} \sigma_2(a_2)^{-1} d^\times a_1 d^\times a_2 dn$$

where

$$n = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}, \quad dn = dx.$$

Now for each $\lambda$, there is a function $\Phi_S[\bullet;\lambda]$ on $E_S^2$ such that, for $g \in G_S$,

$$\int_{E_S\times E_S^\times} f_1\left[n\begin{pmatrix} a_1 & 0 \\ 0 & 1\end{pmatrix}g\right] | a_1 |^{-1/2-\lambda_1} \sigma_1(a_1)^{-1}d^\times a_1 dn = \Phi_S[(0,1)g;\lambda]\sigma_1(\det g) | \det g |^{1/2+\lambda_1} .$$

Since the function $f_1$ has compact support, the support of the function $\Phi_S[\bullet;\lambda]$ is contained in a compact set of $E_S^2 - 0$ which is independent of $\lambda$. Then, for

$g \in G_S$,

$$I_{\lambda,\sigma}(f)\phi(g;\lambda) = \sigma_1(\det g) \mid \det g \mid^{1/2+\lambda_1} \int_{E_S^\times} \Phi_S[(0,t)g;\lambda]\sigma_1\sigma_2^{-1}(t) \mid t \mid^{1+\lambda_1-\lambda_2} d^\times t. \tag{23}$$

To have a formula valid for all $g$ we introduce the characteristic function $\Phi_v :=$ char$(\mathcal{O}_v^2)$ for $v \notin S$ and set

$$\Phi[(x,y);\lambda] := \Phi_S[(x_S,y_S);\lambda] \prod_{v\notin S} \Phi_v[(x_v,y_v)].$$

Then, for $g \in G(F_{\mathbb{A}})$,

$$I_{\lambda,\sigma}(f)\phi(g;\lambda) = \frac{\sigma_1(\det g) \mid \det g \mid^{1/2+\lambda_1}}{L^S(\lambda_1-\lambda_2+1,\sigma_1\sigma_2^{-1})} \int_{E_{,}^\times} \Phi[(0,t)g;\lambda]\sigma_1\sigma_2^{-1}(t) \mid t \mid^{1+\lambda_1-\lambda_2} (t) d^\times t. \tag{24}$$

We need to control $\Phi[\bullet;\lambda]$ in terms of $f$. We take $f$ in a fixed bounded set $\mathcal{B}$. This means that $f = f_S f^S$ as before, with $S$ fixed; the function $f_S$ has support in a fixed set $\Xi$, open and relatively compact. We assume that $K_S\Xi = \Xi K_S = \Xi$; the function $f_S$ is invariant on the right and the left under a compact open normal subgroup $K'$ of $\prod_{v\in S,\text{finite}} K_v$. Thus $f$ is invariant under the subgroup $K" = K'K^S$ of $K^\infty$. For each element $X$ of the enveloping algebra of $G_\infty$ there is a constant $C_X$ such that

$$\sup \mid \rho(X)f \mid \leq C_X.$$

We need not consider those $\sigma$ for which $I_{\lambda,\sigma}(f) = 0$. Thus we may assume that the characters $\sigma_i$ are unramified outside $S$ and that, for $v \in S$ and finite, the restrictions of the characters $\sigma_{i,v}$ to $\mathcal{O}_v^\times$ are in a finite set. The Schwartz–Bruhat function $\Phi_S[\bullet;\lambda]$ is compactly supported: its support is contained in a fixed (i.e. independent of $(\lambda,\sigma)$) open set of $E_S^2 - 0$, relatively compact in $E_S^2 - 0$. The function is invariant on the right under a the action of a fixed compact open subgroup $K_0'$ of $G_S$. Thus its support is contained in a fixed compact set of $E_S^2$ and it is invariant under translation by a compact open subgroup of $E_{S_{\text{finite}}}$. Take $\lambda$ in a strip $a \leq \Re\lambda \leq b$. Then $\Phi_S[\bullet;\lambda]$ is bounded uniformly in $\lambda$ and $\sigma$. Consider a differential operator $\xi$ on $E_\infty^2$ with constant coefficients. On the complement of $(0,0)$ we can write $\xi = \sum_i c_i X_{i,a}$ where the $X_i$ are elements of the enveloping algebra, $X_{i,a}$ are the corresponding vectors fields on $F_\infty^2$ and the functions $c_i$ are smooth functions on $E^2_\infty - (0,0)$. A polynomial on the set of characters of $E^\times_{\mathbb{A}}$ (or $E^\times_\infty$) is a function of the form

$$\chi \mapsto \langle \chi, X \rangle$$

where $X$ belongs to the enveloping algebra of $E^\times_\infty$. It follows that there is a polynomial $P_\xi(\lambda, \sigma_1)$ such that

$$| \xi \Phi_S[(x,y);\lambda] | \leq \| \phi \| \, |P_\xi(\lambda, \sigma_1)|.$$

In particular, for a given $\sigma_1$, and $\lambda$ in a strip, the functions $\Phi_S[(x,y);\lambda,\sigma_1]$ remain in a bounded set of $\mathcal{C}^\infty_c(E^2_S)$ and a fortiori of the Schwartz–Bruhat space $\mathcal{S}(E^2_S)$. Their partial Fourier transform $\hat{\Phi}_S$ with respect to the second variable (see below) are thus also in a bounded set. Moreover, we have in fact more information: for $v$ finite in $S$, the projection of the support of the function $\hat{\Phi}_S$ on $E^2_v$ is contained in a fixed compact set and the function is invariant under a fixed compact open subgroup of $E^2_v$. Finally, for every $\xi$ and every integer $N$ there is a majorization of the form

$$| \xi \hat{\Phi}_S[(x,y);\lambda] | \leq \left(1 + | (x_\infty, y_\infty) |^{\,2}\right)^{-N} |P_{\xi,N}(\sigma_1, \lambda)|.$$

5.2 Whittaker linear form. We define a linear form $\mathcal{W}_{\lambda,\sigma}$ on the space $I(\sigma)$ by

$$\mathcal{W}_{\lambda,\sigma}(\phi) := \int \phi\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] e^{\langle \lambda+\rho, H(*)\rangle} \psi(-x)dx$$

where $w$ is given by (6). It will be convenient for the rest of this section to assume that $\lambda_1 = -\lambda_2$ and to write $\lambda$ for the pair $(\lambda, -\lambda)$. The integral converges for $\Re\lambda >> 0$. It is known that it extends by analytic continuation to the whole complex plane (this follows in fact from the forthcoming considerations). What we need are estimates for $\mathcal{W}_{\lambda,\sigma}\left(I_{\lambda,\sigma}(f)\phi\right)$. Consider again a bounded set $\mathcal{B}$. In what follows, we consider only those $\sigma$ such that $I_{\lambda,\sigma}(f) \neq 0$ for $f \in \mathcal{B}$.

If we use (23) and change variables, we obtain, for $\Re\lambda >> 0$,

$$(25) \quad \mathcal{W}_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi) = \frac{1}{L^S(2\lambda+1, \sigma_1\sigma_2^{-1})} \int \hat{\Phi}[(t,t^{-1});\lambda] \, | \, t \, |^{2\lambda} \, \sigma_1\sigma_2^{-1}(t) d^\times t$$

where $\hat{\Phi}$ denotes the Fourier transform with respect to the second variable:

$$\hat{\Phi}(u,v) = \int \Phi(u,x)\psi(-xv)dx.$$

This expression converges for all $\lambda$. Assuming that $f = f_S f^S$ as before, we can also write this formula in the form

$$\mathcal{W}_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi) = \frac{1}{L^S(2\lambda+1,\sigma_1\sigma_2^{-1})}\int_{F_S^\times}\hat{\Phi}_S[(t,t^{-1});\lambda] \mid t \mid^{2\lambda} \sigma_{1,S}\sigma_{2,S}^{-1}(t)d^\times t. \tag{26}$$

Another way to look at the formula is as follows: Let us assume that $\phi = \otimes\phi_v$ where $\phi_v$ is in the local induced representation $I(\sigma_v)$ and $\phi_v$ is equal to 1 on $K_v$ for $v \notin S$. Then for $f = f_S f^S$, $f_S = \otimes_S f_v$, we get

$$\mathcal{W}_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi) = \frac{1}{L^S(2\lambda+1,\sigma_1\sigma_2^{-1})}\prod_{v\in S}\mathcal{W}_{\lambda,\sigma_v}(I_{\lambda,\sigma_v}(f_v)\phi_v)$$

where the local linear forms $\mathcal{W}_{\lambda,\sigma_v}$, $v \in S$, have similar integral representations:

$$\mathcal{W}_{\lambda,\sigma_v}(I_{\lambda,\sigma_v}(f_v)\phi_v) = \int \hat{\Phi}_v[(t,t^{-1});\lambda] \mid t \mid^{2\lambda} \sigma_{1,v}\sigma_{2,v}^{-1}(t)d^\times t; \tag{27}$$

here the function $\Phi_v$ is determined by $\phi_v$, $f_v$ and $\sigma_{1,v}$. It will be convenient to introduce in the local case or global case the representation $r_2$ of $GL(2)$ on the space of Schwartz–Bruhat functions in two variables defined by

$$r_2(g)\hat{\Phi} = \widehat{g\Phi}, \quad g\Phi(x,y) = \Phi[(x,y)g].$$

Then

$$\mathcal{W}_{\lambda,\sigma_v}(I_{\lambda,\sigma}(g)I_{\lambda,\sigma_v}(f_v)\phi_v) = \int r_2(g)\hat{\Phi}_v[(t,t^{-1});\lambda] \mid t \mid^{2\lambda} \sigma_{1,v}\sigma_{2,v}^{-1}(t)d^\times t\sigma_1(\det g) \mid \det g \mid^{1/2+\lambda_1}.$$

This being so, it is easy to obtain majorizations for the integral

$$\int_{F_S^\times}\hat{\Phi}_S[(t,t^{-1});\lambda] \mid t \mid^{2\lambda} \sigma_{1,S}\sigma_{2,S}^{-1}(t)d^\times t$$

in a vertical strip. We have then estimates for the derivatives as well. Alternatively, for a given $S$, we may introduce a normalized functional $\mathcal{W}_\lambda^1$ on the space of $K^S$-invariant vectors. It is defined by

$$\mathcal{W}_\lambda(\phi) = \frac{1}{L^S(2\lambda+1,\sigma_1\sigma_2^{-1})}\mathcal{W}_\lambda^1(\phi).$$

The preceeding discussion gives majorizations for $\mathcal{W}_\lambda^1(I_\lambda(f_S)\phi)$:

PROPOSITION 4: *Given a bounded set $\mathcal{B}$ there is a polynomial $P(\lambda,\sigma)$ such that, for all $f \in \mathcal{B}$ and all $\lambda$ with $\Re\lambda = 0$,*

$$\left|\mathcal{W}^1_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi)\right| \leq |P(\lambda,\sigma)||\,\phi| \,.$$

*Moreover, given $f$ and $\phi$, the derivatives of $\lambda \mapsto \mathcal{W}^1_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi)$ are of slow increase on the line $\Re\lambda = 0$.*

Later, we will use estimates for $L$-functions that we state as a lemma.

LEMMA 7: *Let $S$ be a finite set of places containing all the places at infinity. For $v$ finite in $S$ let $\Xi_v$ be a finite set of characters of $\mathcal{O}_v^\times$. Let $\Xi$ be the set of idele class characters $\chi$ of $E$ which are trivial on $\mathbb{R}_+^\times$, are unramified outside $S$ and such that, for $v$ finite in $S$, the restriction of $\chi_v$ to $\mathcal{O}_v^\times$ belongs to $\Xi_v$. Then there is a polynomial $P(t,\chi)$ such that for $\chi \in \Xi$ and $t$ real*

$$\left|\frac{1}{L^S(1+it,\chi)}\right| \leq |P(t,\chi)|,$$

*and all $\chi$ non-trivial*

$$\left|L^S(1+it,\chi)\right| \leq |P(t,\chi)|.$$

*Moreover, all derivatives of the functions*

$$t \mapsto \frac{1}{L^S(1+it,\chi)} \quad \text{and} \quad t \mapsto L^S(1+it,\chi)$$

*are of slow increase.*

For a fixed $\chi$, the proof is standard (see, for instance, [L]). The proof can be extended to the general case. ∎

Combining the two previous lemmas, we obtain majorizations for the linear forms $\mathcal{W}_{\lambda,\sigma}$.

5.3 INTERTWINING PERIOD INTEGRAL. We now assume that $E = F[\sqrt{\delta}]$. We assume that the two characters $\sigma_1$ and $\sigma_2$ are related by $\sigma_2(a) = \sigma_1(\overline{a})^{-1}$. Then $\sigma_1\sigma_2^{-1}(a) = \sigma_1|F \circ \mathrm{Norm}$. We let $T$ be the torus of $GL(2,F)$ whose elements have the form

$$t = \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}.$$

We let $\eta \in GL(2,E)$ be an element such that

$$\eta T \eta^{-1} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \right\}$$

and consider the **intertwining period integral** (cf. [JLR])

$$J(\xi,\phi;\lambda,\sigma):=\int_{T(F_{\mathbb{A}})\backslash GL(2,F_{\mathbb{A}})}\phi(\eta h)e^{\langle\lambda+\rho,H(\eta h)\rangle}dh.$$

The integral converges for $\Re\lambda >> 0$. As before if we use the integral representation (24) we obtain a formula for $J(\xi,I_{\lambda,\sigma}(f)\phi;\lambda,\sigma)$:

$$J(\xi,I_{\lambda,\sigma}(f)\phi;\lambda,\sigma):= \tag{28}$$
$$\frac{1}{L^S(2\lambda+1,\sigma_1|F\circ\mathrm{Norm})}\times\int_{GL(2,F_{\mathbb{A}})}\Phi[(0,1)\eta h;\lambda]\sigma_1(\det h)\mid\det|_F^{1+2\lambda}\,dh.$$

Equivalently, we define a Schwartz–Bruhat function $\Psi[\bullet;\lambda]$ on $F_{\mathbb{A}}^2$ (which depends on $\lambda$) in the following way:

$$\Psi[(a_1,a_2);\lambda]:=\int_{K\cap GL(2,F_{\mathbb{A}})}\int_{F_{\mathbb{A}}}\Phi\left[(0,1)\eta\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix}k;\lambda\right]dx\sigma_1(\det k)dk.$$

Then

$$J(\xi,I_{\lambda,\sigma}(f)\phi;\lambda,\sigma)=\frac{1}{L^S(2\lambda+1,\sigma_1|F\circ\mathrm{Norm})} \tag{29}$$
$$\times\int_{(F_{\mathbb{A}}^\times)^2}\Psi[(a_1,a_2);\lambda]\sigma_1(a_1)\mid a_1|_F^{2\lambda}\,\sigma_1(a_2)\mid a_2|_F^{2\lambda+1}\,d^\times a_1d^\times a_2.$$

If we denote by $\tilde{}$ the Fourier transform with respect to the first variable and use the Tate functional equation we get

$$J(\xi,I_{\lambda,\sigma}(f)\phi;\lambda,\sigma)=\frac{1}{L^S(2\lambda+1,\sigma_1|F\circ\mathrm{Norm})} \tag{30}$$
$$\times\int\tilde{\Psi}[(a_1,a_2);\lambda]\sigma_1^{-1}(a_1)\mid a_1|_F^{1-2\lambda}\,\sigma_1(a_2)\mid a_2|_F^{2\lambda+1}\,d^\times a_1d^\times a_2.$$

In fact, with the previous notations and assumptions, we have

$$\tilde{\Psi}=\tilde{\Psi}_{S_0}\prod_{v\notin S_0}\mathrm{char}(\mathcal{O}_{v0}^2)$$

where $\Psi_{S_0}$ is a suitable Schwartz–Bruhat function. Moreover

$$L^S(2\lambda+1,\sigma_1|F\circ\mathrm{Norm})=L^{S_0}(2\lambda+1,\sigma_1|F)L^{S_0}(2\lambda+1,\omega_{E/F}\sigma_1|F).$$

Thus we can get a more explicit formula:

$$\frac{L^{S_0}(1-2\lambda,\sigma_1^{-1}|F)}{L^{S_0}(1+2\lambda,\omega_{E/F}\sigma_1|F)} \tag{31}$$
$$\times\int\int_{F_{S_0}}\tilde{\Psi}_{S_0}[(a_1,a_2);\lambda]\sigma_{1,S_0}^{-1}(a_1)\mid a_1|_F^{1-2\lambda}\,\sigma_{1,S_0}(a_2)\mid a_2|_F^{2\lambda+1}\,d^\times a_1d^\times a_2.$$

In particular, the above expression is a meromorphic function of $\lambda$; on the line $\Re\lambda = 0$, the only singularity is a simple pole at $\lambda = 0$, if the restriction of $\sigma_1$ to $F$ is trivial. This expression can be used to construct a linear form (depending meromorphically on $\lambda$) on the space $I(\sigma)$ (smooth vectors). Indeed, let us denote (31) by $\beta(f, \phi; \lambda)$. Thus $\lambda \mapsto \beta(f, \phi; \lambda)$ is a meromorphic function of $\lambda$ with values in the space of bilinear forms on $\mathcal{C}_C^\infty(G(E_{\mathbb{A}})) \times I(\sigma)$. It satisfies the identity

$$\beta(f_1, I_{\lambda,\sigma}(f_2)\phi; \lambda) = \beta(f_1 * f_2, \phi; \lambda),$$

because for $\Re\lambda >> 0$ the bilinear form is given by a convergent integral and the equality is then obvious. The construction shows that for a given $\lambda$ the bilinear form is separately continuous. In fact if $f = f_S f^S$ and $\phi$ is $K^S$ invariant so that $\phi = \phi_S \phi^S$, we may write

$$\beta(f, \phi; \lambda) := \frac{L^{S_0}(1 - 2\lambda, \sigma_1^{-1}|F)}{L^{S_0}(1 + 2\lambda, \omega_{E/F}\sigma_1|F)} \times \beta_S(f_S, \phi_S; \lambda)$$

where $\lambda \mapsto \beta_S(f_S, \phi_S; \lambda)$ is a meromorphic function with values in the space of bilinear forms on $\mathcal{C}_C^\infty(G(E_S)) \times I(\sigma_S)$ given by

$$\beta_S(f_S, \phi_S; \lambda) =$$
$$\int\int_{F_{S_0}} \tilde{\Psi}_{S_0}[(a_1, a_2); \lambda]\sigma_{1,S_0}^{-1}(a_1) \mid a_1 \mid_F^{1-2\lambda} \sigma_{1,S_0}(a_2) \mid a_2 \mid_F^{2\lambda+1} d^\times a_1 d^\times a_2.$$

We use this to prove a lemma:

LEMMA 8: *Suppose there are $n$ elements $\phi_i$, $1 \le i \le n$ of $I(\sigma_S)$ and $n$ smooth functions of compact support $f_i$ on $G_S$ such that*

$$\sum_{1 \le i \le n} I_{\lambda,\sigma_S}(f_i)\phi_i = 0;$$

*then*

$$\sum_{1 \le i \le n} \beta_S(f_i, \phi_i; \lambda) = 0.$$

*Proof:* Let $f_\alpha$ be an approximation of unity on the group $G_S$. We have

$$\sum \beta_S(f_\alpha, \phi_i; \lambda) = \beta_S\left(f_\alpha, \sum I_{\lambda,\sigma}(f_i)\phi_i; \lambda\right) = 0.$$

If we let $\alpha$ tend to infinity, we get $f_\alpha * f_i \to f_i$; the separate continuity of $\beta_S$ implies the lemma.

We may then define a linear form $J_S(\xi, \bullet; \lambda, \sigma)$ on $I(\sigma_S)$ in the following way: given $\phi$, we use the Dixmier Malliavin lemma to write

$$\phi = \sum I_{\lambda,\sigma_S}(f_i)\phi_i$$

and then set

$$J_S(\xi, \phi; \lambda, \sigma) = \sum \beta(f_i, \phi_i; \lambda).$$

By the lemma, the right hand side depends only on $\phi$ so that the linear form is well defined. In other words, the linear form is uniquely determined by the condition that

$$J_S(\xi, I_{\lambda,\sigma_S}(f_S)\phi; \lambda, \sigma) = \beta(f_S, \phi; \lambda).$$

For a given $\lambda$ it is bounded on any bounded set of $I(\sigma_S)$. The functions in a bounded set of $I(\sigma_S)$ are invariant under a fixed open subgroup of $K_v$, for each finite $v \in S$; for each element $X$ of the enveloping algebra of $K_\infty$ there is a constant $C_X$ such that

$$| \rho(X)\phi(k) | \leq C_X.$$

We take for granted that we can write the functions in a bounded set in the form

$$\phi = \sum_{1 \leq i \leq n} I_{\lambda,\sigma}(f_i)\phi_i,$$

with $n$ fixed, each $f_i$ in a bounded set and each $\phi_i$ in a bounded set. Since

$$J_S(\xi, \phi; \lambda, \sigma) = \sum_i \beta_S(f_i, \phi_i)$$

our assertion follows. As a matter of fact, one can prove that the above decomposition is possible with highly differentiable functions rather than $C^\infty$ functions (cf. [A1], Corollary 4.2); this is good enough for the above considerations and the assertion follows.

It will be convenient to introduce a normalized intertwining period integral $J^1$. For $\phi$ invariant under $K^S$, and $f$ as before, we have

$$J(\xi, I_{\lambda,\sigma}(f)\phi; \lambda, \sigma) = \frac{L^{S_0}(1-2\lambda, \sigma_1^{-1}|F)}{L^{S_0}(1+2\lambda, \omega_{E/F}\sigma_1|F)} \times J^1(\xi, I_{\lambda,\sigma}(f_S)\phi; \lambda, \sigma).$$

It is then easy to obtain majorizations:

PROPOSITION 5: *The normalized period $J^1$ has no singularity on the line $\Re\lambda = 0$. Let $\mathcal{B}$ be a bounded set of smooth functions of compact support. Then, there is* *a polynomial $P(\lambda,\sigma_1)$ such that for any $f$ in $\mathcal{B}$, any $\phi \in I(\sigma)$, and any $\sigma_1$ whose restriction to $F$ is non-trivial,*

$$\left|J^1(\xi, I_{\lambda,\sigma}(f)\phi;\lambda,\sigma)\right| \leq | P(\lambda,\sigma_1)| \ \| \phi \| .$$

*Likewise, any derivative of $\lambda \mapsto J^1(\xi, I_{\lambda,\sigma}(f)\phi;\lambda,\sigma)$ on the line $\Re\lambda = 0$ is of slow increase.*

Similar estimates are available for $J$ in the case where $\sigma_1$ has a trivial restriction to $F$.

5.4 INTERTWINING OPERATOR. We can use the same ideas to obtain majorizations for the matrix coefficients of the intertwining operator, that is, the functions of the form

$$\langle M(w,\lambda)I_{\lambda,\sigma}(f)\phi, \phi'\rangle$$

with $\phi \in I(\sigma)$ and $\phi' \in I(w\sigma)$. Indeed

$$M(w,\lambda)I_{\lambda,\sigma}(f)(g;\lambda) = \int I_{\lambda,\sigma}(f)\phi(wn;\lambda)dn.$$

Using (24) and a change of variables, we get this is the inverse of the appropriate $L$-factor times

$$\int \Phi[(t,x)g;\lambda] \mid \det t \mid^{x\lambda} \sigma_1\sigma_2(t)^{-1} d^\times t \sigma_1(\det g) \mid \det g \mid^{\lambda+1/2} .$$

We introduce the symplectic Fourier transform

$$\hat{\Phi}(x,y) = \int\int \Phi(u,v)\psi(xv-yu)dudv.$$

Using the Tate functional equation we get this is

$$\int \hat{\Phi}[(0,t)g;\lambda] \mid \det t \mid^{1-2\lambda} \sigma_2\sigma_1(t)^{-1} d^\times t \sigma_2(\det g) \mid \det g \mid^{-\lambda+1/2} .$$

Finally, with the previous notation and assumptions, we get, for $g \in G_S$,

$$M(w,\lambda)I_{\lambda,\sigma}(f)(g;\lambda) = \frac{L^S(1-2\lambda,\sigma_2\sigma_1^{-1})}{L^S(1+2\lambda,\sigma_1\sigma_2^{-1})}$$

$$\times \int \hat{\Phi}_S[(0,t)g;\lambda] \mid \det t \mid^{1-2\lambda} \sigma_{2,S}\sigma_{1,S}^{-1}(t) d^\times t \sigma_{2,S}(\det g) \mid \det g \mid^{-\lambda+1/2} .$$

It is then easy to derive the required majorizations.

5.5 WHITTAKER FUNCTIONAL FOR $GL(3)$. In the remainder of this section, we let $G$ be the group $GL(3)$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be a triple of idele class characters of $E$. They are normalized by the condition that they be trivial on $\mathbb{R}_+^\times$. We consider the space $I(\sigma)$ of smooth functions $\phi$ such that

$$\phi\left[\begin{pmatrix} a_1 & x & z \\ 0 & a_2 & y \\ 0 & 0 & a_3 \end{pmatrix} g\right] = \sigma_1(a_1)\sigma_2(a_2)\sigma_3(a_3)\phi(g).$$

As usual, we have a holomorphic fiber bundle of induced representations $I_{\lambda,\sigma}$. Here we let $\lambda$ be a triple of complex numbers $(\lambda_1, \lambda_2, \lambda_3)$ (with $\sum \lambda_i = 0$). Then we set

$$e^{\langle \lambda+\rho, H(g)\rangle} = | a_1 |^{1+\lambda_1} | a_2 |^{\lambda_2} | a_3 |^{-1+\lambda_3}$$

if $g = nak$ and $a = \mathrm{diag}(a_1, a_2, a_3)$. If $\phi$ is in $I_\sigma$, then the function $\phi(\bullet, \lambda)$ defined by

$$\phi(g; \lambda) = \phi(g)e^{\langle \lambda+\rho, H(g)\rangle}$$

is a standard section. To a standard section we can associate an Eisenstein series

$$E(g, \phi; \lambda, \sigma) = \sum \phi(\gamma g; \lambda)e^{\langle \lambda+\rho, H(\gamma g)\rangle}$$

and then set

$$\mathcal{W}_{\lambda,\sigma}(\phi) = \int E(n, \phi; \lambda, \sigma)\overline{\theta}(n)dn.$$

We define in this way a linear form $\mathcal{W}_{\lambda,\sigma}$ on the space of $K$-finite vectors of $I_\sigma$. Equivalently, $\mathcal{W}_{\lambda,\sigma}$ is obtained by analytically continuing the integral

$$\int \phi(wn; \lambda)\overline{\theta}(n)dn,$$

where $w$ is given by (6). In fact it is known that the linear form is defined on the space of smooth vectors. If $\phi = \prod_v \phi_v$ where, for $v \notin S$, $\phi_v$ is equal to 1 on $K_v$, then we can write

$$\mathcal{W}_{\lambda,\sigma}(\phi) = \frac{1}{\prod_{i<j} L^S(1+\lambda_i - \lambda_j, \sigma_i\sigma_j^{-1})} \prod_{v\in S} \mathcal{W}_{\lambda,\sigma_v}(\phi_v),$$

where $\mathcal{W}_{\lambda,\sigma_v}$ is a local Whittaker linear form: the analytic properties of the local linear form are known (see the references to Casselman, Shahidi and Wallach), in particular, the fact that it is a continuous linear form on the space of smooth vectors. Again, we can introduce a normalized linear form on the space of $K^S$invariant vectors by

$$\mathcal{W}_{\lambda,\sigma}(\phi) = \frac{1}{\prod_{i<j} L^S(1+\lambda_i - \lambda_j, \sigma_i\sigma_j^{-1})} \mathcal{W}^1_{\lambda,\sigma}(\phi).$$

We need estimates uniform in $\lambda$ and $\sigma$ for $\mathcal{W}^1_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi)$.

PROPOSITION 6: *Let $\mathcal{B}$ be a bounded set of the space of smooth functions of compact support. There is a polynomial $P(\lambda,\sigma)$ such that for $f_i \in \mathcal{B}$, $i = 1,2$, for any $\sigma$, and any $\phi \in \sigma$, any $\lambda$ with $\Re\lambda = 0$,*

$$\left|\mathcal{W}^1_{\lambda,\sigma}(I_{\lambda,\sigma}(f_1 * f_2)\phi)\right| \leq |P(\lambda,\sigma)||\ \phi| \ .$$

We can also show that the derivatives of such a function are of slow increase on the space $\Re\lambda = 0$. We will sketch a proof of these estimates below. Combining with standard estimates on Abelian $L$-functions we will obtain similar estimates for $\mathcal{W}_{\lambda,\sigma}$.

Now we discuss how to obtain estimates for the local linear forms. Thus for the remaining part of this section we let $E$ be a local field. We have then the local representation $I(\sigma)$. The proof is again based on an integral representation for a convolution section and the value of the linear form on a convolution section. We will simply write down the integrals in question and will leave out the details of the rest of the proof. We let $f_1, f_2$ be two smooth functions of compact support. We find an absolutely convergent expression for $\mathcal{W}_{\lambda,\sigma}(I_{\lambda,\sigma}(f_2 * f_1)\phi)$. We first find an absolutely convergent expression for the section $I_{\lambda,\sigma}(f_2 * f_1)\phi(g;\lambda)$. In a precise way, we set

$$\phi(g;\lambda) = \phi(g)e^{\langle\lambda+\rho,H(g)\rangle},$$

$$\phi_1(g;\lambda) = \int \phi(gx;\lambda)f_1(x)dx, \quad \phi_2(g;\lambda) = \int \phi_1(gx;\lambda)f_2(x)dx.$$

It will be convenient to introduce a function $\phi_1(\bullet,\bullet;\lambda)$ on $GL(3)\times GL(2)$ such that

$$\phi_1(g,e;\lambda) = \phi_1(g;\lambda),$$

$$\phi_1\left(\begin{pmatrix} a_1 & X \\ 0 & g_2 \end{pmatrix} g, x;\lambda\right) = \phi_1(g, xg_2;\lambda) \mid a_1 \mid^{1+\lambda_1} \sigma_1(a_1) \mid \det g_2 \mid^{-1/2} .$$

From the computations on $GL(2)$, it can be seen that there is a Schwartz–Bruhat function $\Phi_1(\bullet,\bullet;\lambda)$ on $E^2\times K$, where $K$ is the standard maximal compact subgroup of $GL(3,E)$ such that

$$\phi_1[k,x;\lambda] =$$
$$\sigma_2(\det x) \mid \det x \mid^{1/2+\lambda_2} \int_{E^\times} \Phi_1[(0,g_1)x;\lambda]\sigma_2\sigma_3^{-1}(g_1) \mid g_1 \mid^{1+\lambda_2-\lambda_3} d^\times g_1.$$

We have then

$$\phi_2(g;\lambda) = \int \phi_1(x;\lambda)\check{f}_2(x^{-1}g)dx.$$

At this point we use the Iwasawa decomposition for the parabolic subgroup of type $(1,2)$ to get that $\phi_2(g;\lambda)$ is equal to

$$\int \phi_1(k_3, g_2; \lambda)\check{f}_2\left(k_3^{-1}\begin{pmatrix}1 & X\\ 0 & 1_2\end{pmatrix}\begin{pmatrix}a_1^{-1} & 0\\ 0 & g_2^{-1}\end{pmatrix}g\right)$$
$$\times\sigma_1(a_1) \mid a_1 \mid^{1+\lambda_1} d^\times a_1 \mid \det g_2 \mid^{-1/2} dg_2.$$

Next, we define a function $\Phi_2(\bullet,\bullet;\lambda)$ on $E^2 \times GL(3,E)$ by

$$\Phi_2[(x,y),g;\lambda] = \int \Phi_1[(x,y),k_3;\lambda]\check{f}_2(k_3^{-1}g)dk_3.$$

The projection of the support of this function on the second variable is contained in a fixed compact set. Now there is a function $\Phi_3(\bullet,\bullet;\lambda)$ on $E^2 \times M(2\times 3, F)$ such that

$$\Phi_3\left[(x,y),(0,1_2)g;\lambda\right]\sigma_1(\det g_1) \mid \det g \mid^{1+\lambda_1} =$$
$$\int \Phi_2\left[(x,y),\begin{pmatrix}1 & X\\ 0 & 1_2\end{pmatrix}\begin{pmatrix}a_1^{-1} & 0\\ 0 & 1_2\end{pmatrix}g\right]\sigma_1(a_1) \mid a_1 \mid^{1+\lambda_1} d^\times a_1.$$

In terms of $\Phi_3$ we can write

(32)
$$\phi_2(g;\lambda) = \sigma_1(\det g) \mid \det g \mid^{1+\lambda_1}$$
$$\times\int \Phi_3\left[(0,g_1)g_2^{-1},(0,g_2)g;\lambda\right]$$
$$\sigma_2\sigma_3^{-1}(g_1) \mid g_1 \mid^{1+\lambda_2-\lambda_3} d^\times g_1 \sigma_1\sigma_2^{-1}(\det g_2) \mid \det g_2 \mid^{1+\lambda_1-\lambda_2} d^\times g_2.$$

We can now write that

$$\mathcal{W}_\lambda(I_\lambda(f_2 * f_1)\phi) = \int \phi_2(wn)\overline{\theta}(n)dn$$
$$= \int \phi_2\left[w_2\begin{pmatrix}1&0&0\\0&1&x\\0&0&1\end{pmatrix}w_1w_2\begin{pmatrix}1&0&z\\0&1&y\\0&0&1\end{pmatrix}\right]\psi(-x)\psi(-y)dxdy.$$

Taking these integrals inside the integral representation leads to an inner integral of the form

$$\int \Phi\left[(g_1,g_1x),\left(g_2,g_2\begin{pmatrix}z\\y\end{pmatrix}\right);\lambda\right]\psi(-x-y)dxdydz.$$

We introduce a partial Fourier transform of $\Phi_4$:

$$\Phi_5\left[(\bullet,u),\begin{pmatrix}\bullet & \bullet & a\\ \bullet & \bullet & b\end{pmatrix};\lambda\right] =$$
$$\int \Phi_4\left[(\bullet,x),\begin{pmatrix}\bullet & \bullet & z\\ \bullet & \bullet & y\end{pmatrix};\lambda\right]\psi(-xu-za-yb)dydz.$$

Recall that the right multiplication of elements of $E^2$ by elements of $GL(2)$ produces a representation of $GL(2)$ on the space of Schwartz–Bruhat functions on $E^2$; by Fourier transform, it produces a new representation denoted $r_2$. Likewise, multiplication of elements of $M(2\times 3, E)$ by elements of $GL(2)$ on the left produces a right representation on the space of Schwartz–Bruhat functions on $M(2\times 3, E)$; by Fourier transform we get a right representation of $GL(2)$ denoted by $l_2$. Thus we have also right and left representations of $GL(2)$ on the space of Schwartz–Bruhat functions on $E^2 \times M(2\times 3, E)$. We denote them in the same way. We obtain finally

$$\int r_2(g_2)^{-1}\Phi_5\left[(g_1, g_1^{-1}), \left(g_2, {}^t g_2^{-1}\begin{pmatrix}0\\1\end{pmatrix}\right);\lambda\right]$$
$$\sigma_2\sigma_3^{-1}(g_1) \mid g_1 \mid^{\lambda_2-\lambda_3} d^\times g_1 \sigma_1\sigma_2^{-1}(\det g_2) \mid \det g_2 \mid^{\lambda_1-\lambda_2} d^\times g_2.$$

We use the Iwasawa decomposition to compute the integral over $GL(2)$:

$$g_2 = k_2\begin{pmatrix} a_1 & x \\ 0 & a_2\end{pmatrix}, \quad dg_2 = dk_2 d^\times a_1 \mid a_2 \mid^{-1} d^\times a_2.$$

We introduce a new Schwartz–Bruhat function

$$\Phi_6 = \int r_2(k_2)^{-1}\Phi_5 l(k_2) dk_2.$$

Then we get

$$\int \Phi_6\left[(a_1^{-1}g_1, a_2 g_1^{-1}), \begin{pmatrix} a_1 & x & 0 \\ 0 & a_2 & a_2^{-1}\end{pmatrix};\lambda\right]$$
$$\sigma_2\sigma_3^{-1}(g_1) \mid \det g_1 \mid^{\lambda_2-\lambda_3} d^\times g_1 \sigma_1\sigma_2^{-1}(a_1a_2) \mid a_1 a_2 \mid^{\lambda_1-\lambda_2} d^\times a_1 d^\times a_2 \psi(-x a_1^{-1}) dx.$$

We introduce finally the Fourier transform $\Phi$ of $\Phi_6$ defined by

$$\Phi\left[(\bullet,\bullet), \begin{pmatrix} \bullet & a & \bullet \\ \bullet & \bullet & \bullet\end{pmatrix};\lambda\right] = \int \Phi_6\left[(\bullet,\bullet), \begin{pmatrix} \bullet & x & \bullet \\ \bullet & \bullet & \bullet\end{pmatrix};\lambda\right]\psi(-xa)dx.$$

Our last expression is then

$$\mathcal{W}_\lambda(I_\lambda(f_1 * f_2)\phi) = \tag{33}$$
$$\int \Phi\left[(a_1^{-1}g_1, a_2 g_1^{-1}), \begin{pmatrix} a_1 & a_1^{-1} & 0 \\ 0 & a_2 & a_2^{-1}\end{pmatrix};\lambda\right]$$
$$\sigma_2\sigma_3^{-1}(g_1) \mid g_1 \mid^{\lambda_2-\lambda_3} d^\times g_1 \sigma_1\sigma_2^{-1}(a_1a_2) \mid a_1 a_2 \mid^{\lambda_2-\lambda_3} d^\times a_2 d^\times a_3.$$

The previous formal manipulations are justified for $\Re\lambda_1 > \Re\lambda_2 > \Re\lambda_3$. The above integral converges for all $\lambda$. Thus the above formula is actually true for all $\lambda$. It can be used to obtain the required majorizations. It could also be used to analytically continue $\mathcal{W}_\lambda$.

## 6. Limit formula for $J_\chi(f)$

We now discuss the continuous part of the spectrum. We follow the notations of [JLR]. Thus we let $E/F$ be a quadratic extension of number fields. For the time being, let $G$ be the group $GL(n)$ regarded as an algebraic group over $E$ and $H$ the group $GL(n)$ regarded as an algebraic group over $F$. We denote by $N$ the group of upper triangular matrices with unit diagonal and by $\theta$ a generic character of $N(E_{\mathbb{A}})$ trivial on $N(E)$. We let $f$ be a smooth function of compact support on $G(\mathbb{A})$. We define as usual a geometric kernel by (2). Let $\chi$ be a cuspidal pair, that is, an equivalence class of pairs $(M,\pi)$ where $M$ is a standard Levi-subgroup of $G$ and $\pi$ is a cuspidal representation of $M$. We assume that the restriction of the central character of $\pi$ to $Z(E_{\mathbb{A}})$ is $\omega$. We let $K_{f,\chi}$ be the corresponding kernel component. With this notation,

$$K_f(x,y) = \sum_\chi K_{f,\chi}(x,y).$$

In addition we define

$$K^0_{f,\chi}(x,y) := \frac{1}{n(\chi)} \sum_{(M,\pi)} \int_{i\mathfrak{A}^*_P} \sum_{\phi\in\mathcal{B}_P(\pi)} E(x, I_\lambda(f)\phi;\lambda)\overline{E(y,\phi;\lambda)}d\lambda.$$

Here $d\lambda$ is an appropriate Haar measure on $i\mathfrak{A}^*_P$. The outer sum is over all pairs $(M,\pi)$ in the class $\chi$. For a given pair, the Eisenstein series at hand is induced from the standard parabolic subgroup $P$ with Levi-factor $M$ and $\mathcal{B}_P(\pi)$ is an orthonormal basis for the induced representation. The integer $n(\chi)$ is the number of chambers in $\mathfrak{a}_P$. Note that the full kernel $K_{f,\chi}$ has an analogous expression which involves Eisenstein series built out of residual forms. It follows from the functional equation of the Eisenstein series that, for a given $\chi$, all the kernels

$$(x,y) \mapsto \int_{i\mathfrak{A}^*_P} \sum_{\phi\in\mathcal{B}_P(\pi)} E(x, I_\lambda(f)\phi;\lambda)\overline{E(y,\phi;\lambda)}d\lambda$$

are actually equal. Thus, if we call $n_1(\chi)$ the number of pairs in the given class $\chi$, and choose one representative pair $(M,\pi)$ in the class, we can also write

$$K^0_{f,\chi}(x,y) = \frac{n_1(\chi)}{n(\chi)} \int_{i\mathfrak{A}^*_P} \sum_{\{\phi\}} E(x, I_\lambda(f)\phi;\lambda)\overline{E(y,\phi;\lambda)}d\lambda.$$

LEMMA 9: *For all $y$,*

$$\int K_{f,\chi}(y,n)\theta(n)dn = \int K^0_{f,\chi}(y,n)\theta(n)dn.$$

*Proof:* This follows from the fact that residual Eisenstein series are degenerate ([MW]). ∎

In most of this paper, we will assume that $f = f_1 * f_2^*$ where each function $f_i$ is $K$-finite. Then

$$K^0_{f,\chi}(x,y) = \frac{n_1(\chi)}{n(\chi)} \sum \int_{i\mathfrak{A}_P^*} \sum_{\{\phi\}} E(x, I_\lambda(f_1)\phi;\lambda)\overline{E(y, I_\lambda(f_2)\phi;\lambda)}d\lambda.$$

We recall some consequences of Arthur's majorizations ([A1]). Suppose $f^\infty$ is a smooth function of compact support on the group $G^\infty$. Then, there is a continuous semi-norm $\mu$ on the space of smooth functions of compact support on $G_\infty$ such that for any $X, Y$ in the enveloping algebra of $G_\infty$ and any $f$ of the form $f = f_\infty f^\infty$ with $f_\infty$ $K_\infty$-finite,

$$(34) \qquad \sum_\chi \frac{n_1(\chi)}{n(\chi)} \int_{i\mathfrak{A}_P^*} \left| \sum_{\phi \in \mathcal{B}_P(\pi)} \rho(X)E(x, I_\lambda(f)\phi;\lambda)\overline{\rho(Y^*)E(y,\phi;\lambda)} \right| d\lambda$$
$$\leq \mu(X * f_\infty * Y) \parallel x \parallel^N \parallel y \parallel^N .$$

Here the action of the enveloping algebra is by infinitesimal right translation: if $X$ is in the Lie algebra of $G_\infty$ and $h$ is a smooth function on $G_\infty$, then

$$\rho(X)h(g) = \left.\frac{dh(g\exp(tX))}{dt}\right|_{t=0} .$$

Similarly

$$(35) \qquad \sum_\chi \left|\rho_1(X)\rho_2(\check{Y})K_{f,\chi}(x,y)\right| \leq \mu(X * f_\infty * Y) \parallel x \parallel^N \parallel y \parallel^N .$$

Here the index 1, for instance, means that the operator $\rho(X)$ is applied to the function $x \mapsto K_{f,\chi}(x,y)$. As usual we denote by $T$ the truncation parameter, by $d(T)$ the minimum of the $\alpha(T)$ for $\alpha \in \Delta_0$. If $N$ is given, we let $G(E_\mathbb{A}, N)$ be the set of $x \in G(E_\mathbb{A})$ such that $\log \parallel x \parallel \leq N$. We denote by $\Lambda^T_m$ the **relative truncation operator** introduced in [JLR]. We denote by $\Lambda^T_{1,m}K(x,y)$ the partial truncation operator, that is, the truncation operator applied to the function $x \mapsto K(x,y)$. We recall a lemma which is based on the work of Arthur (cf. [J3] Proposition 21):

LEMMA 10: *Let $\Omega$ be a compact subset of $G(E_\mathbb{A})$. There is a constant $C$ such that if $f$ has support in $G(E_\mathbb{A}, N)$ then, for $d(T) \geq C(1+N)$, $y \in \Omega$ any $x \in H(F_\mathbb{A})$ and any $\chi$,*

$$\Lambda^T_{1,m}K_f(x,y) = K_f(x,y), \quad \Lambda^T_{1,m}K_{f,\chi}(x,y) = K_{f,\chi}(x,y).$$

Now let $f^\infty$ be given. Let $\Omega$ be an open subset of $G_\infty$ relatively compact and such that $K_\infty \Omega K_\infty = \Omega$. Let also $\Omega_1$ be a compact subset of $G(F_{\mathbb{A}})$. Then there is $C$ such that for $d(T) > C$, $f = f_\infty f^\infty$, with $f_\infty$ supported on $\Omega$, $x \in H(F_{\mathbb{A}})$, $y \in \Omega_1$, and any $\chi$,

$$\Lambda^T_{1,m} K_{f,\chi}(x,y) = K_{f,\chi}(x,y).$$

Since truncation transforms slowly increasing functions into rapidly decreasing functions, we see from (35) that we can truncate the series

$$\sum_\chi K_{f,\chi}(x,y)$$

termwise. Moreover, there is a continuous semi-norm $\mu$ on $\mathcal{C}_c^\infty(\Omega)$ such that, under the same conditions,

$$\sum_\chi |K_{f,\chi}(x,y)| \leq \mu(f_\infty). \tag{36}$$

We apply this to a compact set $\Omega_1$ of $N(E_{\mathbb{A}})$ such that $N(E_{\mathbb{A}}) = \Omega_1 N(E)$. After integrating over $N(E)\backslash N(E_{\mathbb{A}})$ we get

$$\sum_\chi \left| \int K_{f,\chi}(x,n)\theta(n)dn \right| \leq \mu(f_\infty)$$

or

$$\sum_\chi \left| \int K^0_{f,\chi}(x,n)\theta(n)dn \right| \leq \mu(f_\infty).$$

At this point we introduce the spectral distributions to be investigated in this paper:

$$J_\chi(f) = \int_{H(F)Z(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} dh \int_{N(E)\backslash N(E_{\mathbb{A}})} K_\chi(h,n)\theta(n)dn. \tag{37}$$

By the previous majorization this is well defined. Note that in fact $J_\chi(f)$ is equal to

$$\int_{H(F)Z(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} dh \int_{N(E)\backslash N(E_{\mathbb{A}})} K^0_\chi(h,n)\theta(n)dn.$$

From the previous majorization we also get

$$\sum_\chi |J_\chi(f)| \leq \mu(f_\infty) \tag{38}$$

for $f = f_\infty f^\infty$, $f_\infty$ supported on $\Omega$. In particular, for a fixed $\chi$,

$$|J_\chi(f)| \leq \mu(f_\infty). \tag{39}$$

This shows that the linear form $f_\infty \mapsto J_\chi(f)$, defined on $K$-finite functions, extends to a distribution. Thus $J_\chi(f)$ is defined for all $f$. Let $S$ be a finite set of cuspidal data. Consider the inequality

$$\sum_{\chi\in S}|J_\chi(f)| \le \mu(f_\infty).$$

It is true for $f_\infty$ $K$-finite with support on $\Omega$. By continuity, it is then true for any $f_\infty \in \mathcal{C}_c^\infty(\Omega)$. Since $S$ is arbitrary, we see that the above inequality (38) is true for any $f_\infty \in \mathcal{C}_c^\infty(\Omega)$.

To continue we use Lemma 9 to conclude that there is $C > 0$ such that, for $x \in H(F_{\mathbb{A}})$ and $f = f_\infty f^\infty$ with $f_\infty \in \mathcal{C}_c^\infty$, we have, for $d(T) > C$ and any $\chi$,

$$\Lambda_m^T \int K^0_{f,\chi}(x,n)\theta(n)dn = \int \Lambda^T_{1,m}K^0_{f,\chi}(x,n)\theta(n)dn = \int K^0_{f,\chi}(x,n)\theta(n)dn.$$

It follows that

$$J_\chi(f) = \int\int \Lambda^T_{1,m}K^0_{f,\chi}(x,n)\theta(n)dndx.$$

Furthermore, the inequality (34) implies that we can take the truncation and the integrations inside the expression for $K^0$. We introduce

$$\mathcal{W}_{\lambda,\sigma}(\phi) := \int E(n,\phi;\lambda)\overline{\theta}(n)dn.$$

We often drop $\sigma$ from the notation and write simply $\mathcal{W}_\lambda(\phi)$. Then we get

PROPOSITION 7: *Let $f_i^\infty$, $i = 1,2$ be given as well as an open set $\Omega$ of $G_\infty$ relatively compact and bi-$K$ invariant. There is $C$ such that if $f_i = f_{i,\infty}f_i^\infty$, $i = 1,2$, $f_{i,\infty}$ supported on $\Omega$, then*

$$J_\chi(f) = \frac{n_1(\chi)}{n(\chi)}\int_{i\mathfrak{A}_P^*}\sum_{\{\phi\}}\left(\int \Lambda_m^T E(h, I_\lambda(f_1)\phi,\lambda)dh\right)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}d\lambda,$$

*for all $T$ with $d(T) \ge C$.*

Note that the integral is absolutely convergent in the sense that

$$\int_{i\mathfrak{A}_P^*}\left|\sum_{\{\phi\}}\left(\int \Lambda_m^T E(h, I_\lambda(f_1)\phi,\lambda)dh\right)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}\right|d\lambda$$

is finite. We now go back to the case $n = 3$.

## 7. Contribution of the parabolic of type (2,1)

We let $\chi$ be a cuspidal pair of the form $(M,\sigma)$, where $M = GL(2) \times GL(1)$ and $\sigma$ is a cuspidal automorphic representation of $GL(2,E_{\mathbb{A}}) \times GL(1,E_{\mathbb{A}})$. Then $n(\chi) = n_1(\chi)$. By [JLR], the term $J_\chi(f)$ is zero unless $\sigma$ is distinguished, that is, $\sigma = \sigma_1 \otimes \sigma_2$ where $\sigma_1$ is a cuspidal automorphic representation of $GL(2,E_{\mathbb{A}})$ distinguished by $GL(2,F_{\mathbb{A}})$ and $\sigma_2$ an idele class character of $E$ trivial on $F_{\mathbb{A}}^\times$. As usual, $\sigma_1$ is normalized by the condition that its central character be trivial on $\mathbb{R}_+^\times$. Likewise for $\sigma_2$. We let $Q$ be the parabolic subgroup of type $(2,1)$ and $Q'$ the parabolic subgroup of type $(1,2)$. As before we can define the space $I(\sigma)$ of smooth functions $\phi : G(E_{\mathbb{A}}) \to \mathbb{C}$ such that, for any $k \in K$, the function $m \mapsto \phi(mk)$ is in the space of smooth vectors of $\sigma$. Let $\phi$ be a $K$-finite vector in $I(\sigma)$. Consider an Eisenstein series $E(g,\phi;\lambda)$. Then

$$\int \Lambda_m^T E(h,\phi;\lambda)dh = v_Q \frac{e^{\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} J(1,\phi,0) + v_{Q'} \frac{e^{\langle s\lambda,T\rangle}}{\langle s\lambda,\check{\alpha}_{Q'}\rangle} J(1,M(s,\lambda)\phi,0), \tag{40}$$

where $s = w_1 w_2$ and

$$J(1,\phi,0) := \int_{K_H} \int_{M_H(F)\backslash M_H^1} \phi(mk)dkdm;$$

here $M_H$ is the product of $GL(2,F)$ and $GL(1,F)$ and $K_H$ the standard maximal compact subgroup of $H(F_{\mathbb{A}})$. The period integral $J(1,\bullet,0)$ will be also denoted by $J_{0,\sigma}$ or $J_{0,\sigma_1,\sigma_2}$. It is a continuous linear form on $I(\sigma)$. In fact,

$$\langle s\lambda, \check{\alpha}_{Q'}\rangle = \langle \lambda, s^{-1}\check{\alpha}_{Q'}\rangle = -\langle \lambda, \check{\alpha}_{Q'}\rangle.$$

Denoting by $\check{\alpha}_i$ the simple coroots, we take $T$ of the form $T = t\check{\alpha}_1 + t\check{\alpha}_2$ with $t > 0$. Then the projections of $T$ on $\mathfrak{A}_Q$ and $\mathfrak{A}_{Q'}$ respectively have the form: $T_Q\check{\alpha}_Q$, $T_Q\check{\alpha}_{Q'}$ with $T_Q > 0$. Thus $\langle s\lambda, T\rangle = -\langle\lambda, T\rangle$. Hence the above formula can be written

$$\int \Lambda_m^T E(h,\phi;\lambda)dh = v_Q \frac{e^{\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} J(1,\phi,0) - v_Q \frac{e^{-\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} J(1,M(s,\lambda)\phi,0).$$

Since both sides of the formula must be holomorphic at $\lambda = 0$ we find

$$J(1,\phi,0) = J(1,M(s,0)\phi,0). \tag{41}$$

To continue, we set

$$\begin{aligned} d_{+,\phi,\phi'}(\lambda) &= J(1,\phi',0)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}, \\ d_{-,\phi,\phi'}(\lambda) &= J(1,M(s,\lambda)\phi',0)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}. \end{aligned}$$

Then $d_{+,\phi,\phi'}(0) = d_{-,\phi,\phi'}(0)$ and
(42)

$$J_\chi(f) = v_Q \sum_{\phi,\phi'} \int \left[ \frac{e^{\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} d_{+,\phi,\phi'}(\lambda) - \frac{e^{-\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} d_{-,\phi,\phi'}(\lambda) \right] \langle I_\lambda(f_1)\phi', \phi\rangle d\lambda.$$

To compute the limit for $T \to +\infty$, we use the method of $(G, M)$ families developed by Arthur ([A3]). Here it amounts to writing the integrand as the sum of the following terms:

$$\begin{gathered} e^{\langle\lambda,T\rangle} \frac{d_{+,\phi,\phi'}(\lambda) - d_{+,\phi,\phi'}(0)}{\langle\lambda,\check{\alpha}_Q\rangle} \langle I_\lambda(f_1)\phi', \phi\rangle \\ -e^{-\langle\lambda,T\rangle} \frac{d_{-,\phi,\phi'}(\lambda) - d_{+,\phi,\phi'}(0)}{\langle\lambda,\check{\alpha}_Q\rangle} \langle I_\lambda(f_1)\phi', \phi\rangle \\ + \left( \frac{e^{\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} - \frac{e^{-\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}_Q\rangle} \right) d_{+,\phi,\phi'}(0)\langle I_\lambda(f_1)\phi', \phi\rangle. \end{gathered}$$

In the first term, the quotient is a smooth function, even at $\lambda = 0$. The reciprocal of the denominator is bounded in the complement of a neighborhood of 0, and $\langle I_\lambda(f)\phi', \phi\rangle$ is integrable and square integrable. Now $J(1, \phi, 0)$ is independent of $\lambda$. If $x$ and $y$ are in compact sets we know that

$$\int \left| \sum_\phi E(x, I_\lambda(f_2)\phi; \lambda) \overline{E(y, I_\lambda(f_2)\phi; \lambda)} \right| d\lambda$$

is finite. Integrating over $N \times N$ we conclude that

$$\int \sum_\phi |\mathcal{W}_\lambda(I_\lambda(f_2)\phi)|^2 \, d\lambda$$

is finite as well. Thus $d_{+,\phi,\phi'}$ is square integrable. Hence the first term can be viewed as the Fourier transform evaluated on the projection of $T$ on $\mathfrak{A}_Q$ of an integrable function. Thus it tends to zero as $T$ tends to infinity. Since the intertwining operator is bounded, a similar conclusion applies to the second term. Finally, the last term can be computed as the integral of $d_{+,\phi,\phi'}(0)$ times the Fourier transform of

$$\langle I_\lambda(f_1)\phi', \phi\rangle$$

over the convex hull of the set $T_Q, -T_Q$. Thus it tends to $\langle I_0(f_1)\phi, \phi'\rangle$ as $T$ tends to infinity. We arrive at our result:

$$J_\chi(f) = v_Q \sum_{\phi,\phi'} \langle I_\lambda(f)\phi' \phi\rangle d_{+,\phi,\phi'}(0)$$

or

$$J_\chi(f) = v_Q \sum_{\{\phi\}} J(1, I_0(f_1)\phi, 0)\overline{\mathcal{W}_0(I_0(f_2)\phi)}. \tag{43}$$

On the space of smooth vectors of the representation $I_{0,\sigma_1,\sigma_2}$ we now have two continuous linear forms: the period form $J_{0,\sigma_1,\sigma_2}$ and the Whittaker form $\mathcal{W}_{0,\sigma_1,\sigma_2}$. Using the notation of generalized vectors we see that we can write our result in the form given in the next proposition.

PROPOSITION 8: *For any smooth function of compact support $f$ we have*

$$J_\chi(f) = v_Q \langle I_{0,\sigma_1,\sigma_2}(f)\mathcal{W}_{0,\sigma_1,\sigma_2}, J_{0,\sigma_1,\sigma_2}\rangle.$$

We have established the result for a $K$-finite function. Since both sides are distributions the identity is true for all functions.

## 8. Contribution of the Borel subgroup

Suppose that $\chi$ is represented by the pair $(A, \sigma)$ where $A$ is the group of diagonal matrices and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. As before, the characters are normalized by the condition that they be trivial on the group $\mathbb{R}_+^\times$. As before, $\lambda$ denotes an element in $\mathfrak{A}^*$. We introduce the complex variables $s_1, s_2, s_3$ defined by

$$e^{\langle \lambda, H(a)\rangle} = | a_1 |^{s_1} | a_2 |^{s_2} | a_3 |^{s_3} .$$

We define the space $I(\sigma)$. Let $\phi$ be $K$-finite in $I(\sigma)$. Consider the corresponding Eisenstein series $E(h, \phi, \lambda)$.

According to [JLR], Theorem 40, the period of the truncated Eisenstein series $\int \Lambda_m^T E(h, \phi, \lambda)dh$ is equal to a sum of the terms indexed by the standard parabolic subgroups of $G$ whose Levi-factors are isomorphic to products of the groups $GL(1)$ and $GL(2)$. In our case,

$$\int \Lambda_m^T E(h, \phi, \lambda)dh = \mathrm{I}(B, \phi, \lambda) + \mathrm{I}(Q, \phi, \lambda) + \mathrm{I}(Q', \phi, \lambda),$$

where $Q$ and $Q'$ are the standard parabolic subgroups of type $(2, 1)$ and $(1, 2)$ respectively, while $B$ is, as before, the standard Borel subgroup. The term attached to $B$ vanishes unless the characters $\sigma_i$ are all distinguished. If they are, then

$$\mathrm{I}(B, \phi, \lambda) = v_B \sum_s \frac{e^{\langle s\lambda, T\rangle}}{\prod_{\alpha\in\Delta}\langle s\lambda, \check{\alpha}\rangle} J(1, M(s, \lambda)\phi, 0) \tag{44}$$

where the sum is over all $s$ in the Weyl group and, as before,

$$J(1,\phi,0)=\int_{K_H}\phi(k)dk.$$

The terms attached to $Q$ and $Q'$ vanish unless $\sigma_i(a)=\sigma_j(\overline{a})^{-1}$ for some pair of indices $i,j$ and the third character $\sigma_k$ is distinguished. Assuming this, the terms are as follows:

$$\mathrm{I}(Q,\phi,\lambda)=\sum_w v_Q\frac{e^{\langle (w\lambda)_Q,T\rangle}}{\langle (w\lambda)_Q,\check{\alpha}_Q\rangle}J(\xi_Q,M(w,\lambda)\phi,(w\lambda)_0^Q) \tag{45}$$

and

$$\mathrm{I}(Q',\phi,\lambda)=\sum_w v_{Q'}\frac{e^{\langle (w\lambda)_{Q'},T\rangle}}{\langle (w\lambda)_{Q'},\check{\alpha}_{Q'}\rangle}J(\xi_{Q'},M(w,\lambda)\phi,(w\lambda)_0^{Q'}). \tag{46}$$

Here

$$\xi_Q=w_1,\quad \eta_Q.\overline{\eta}_Q^{-1}=\xi_Q,$$
$$\xi_{Q'}=w_2,\quad \eta_{Q'}.\overline{\eta}_{Q'}^{-1}=\xi_{Q'}.$$

In (45), the sum is over all $w$ such that $w^{-1}\alpha_1>0$ and $w\sigma$ is trivial on the torus $T_Q$ defined by $T_Q=A(E)\cap\eta_QH(F)\eta_Q^{-1}$, that is, the torus of matrices of the form

$$\mathrm{diag}(\alpha,\overline{\alpha},\beta),\quad \alpha\in E,\beta\in F.$$

Likewise in (46), the sum is over all $w$ such that $w^{-1}\alpha_2>0$ and $w\sigma$ is trivial on the torus $T_{Q'}$. The definition of the intertwining period will be recalled below.

Let us write $J_\chi(f)$ as a sum

$$J_\chi(f)=J_\chi^{\min}(f)+J_\chi^{\max}(f)$$

where

$$J_\chi^{\min}(f)=\frac{n_1(\chi)}{n(\chi)}\sum_\phi\int \mathrm{I}(B,I_\lambda(f_1)\phi,\lambda)\overline{W_\lambda(I_\lambda(f_2)\phi)}d\lambda$$

and

$$J_\chi^{\max}(f)=\frac{n_1(\chi)}{n(\chi)}\sum_\phi\int (\mathrm{I}(Q,I_\lambda(f_1)\phi,\lambda)+\mathrm{I}(Q',I_\lambda(f_1)\phi,\lambda))\,\overline{W_\lambda(I_\lambda(f_2)\phi)}d\lambda.$$

8.1 THE TERM $J_\chi^{\min}(f)$. We now compute $\mathrm{I}(B, \phi, \lambda)$. We assume that the characters $\sigma_i$ are distinguished for $i = 1, 2, 3$ since otherwise the term vanishes. As before, we assume that $f_1$ and $f_2$ are themselves convolutions of $K$-finite functions. In particular,

$$f_1 = f_1^1 * f_1^2.$$

Then

$$I_\lambda(f_1)\phi = \sum_{\phi'} \langle I_\lambda(f_1^2)\phi, \phi' \rangle \phi'.$$

Thus we get

$$J_\chi^{\min}(f) = v_B \frac{n_1(\chi)}{n(\chi)} \sum_{\phi,\phi'} \int \sum_s \frac{e^{\langle s\lambda, T\rangle}}{\prod_{\alpha\in\Delta} \langle s\lambda, \check{\alpha}\rangle} A_{\phi,\phi'}(\lambda) B_{\phi,\phi'}(\lambda) d\lambda$$

where we have set

$$B_{\phi,\phi'}(\lambda) = \langle I_\lambda(f_1^2)\phi, \phi' \rangle,$$
$$A_{\phi,\phi'}(\lambda) = J(1, M(s,\lambda) I_\lambda(f_1^1)\phi', 0) \overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}.$$

As before, the function $\lambda \mapsto B_{\phi,\phi'}(\lambda)$ is of rapid decrease as well as all its derivatives, while

$$\lambda \mapsto A_{\phi,\phi'}(\lambda)$$

is of slow increase as well as all its derivatives.

We shall make use of the theory of $(G, M)$ families as developed by Arthur. Set

$$c_s(\lambda) = e^{\langle s\lambda, T\rangle} \quad \text{and} \quad d_s(\lambda) = A_{\phi,\phi'}(\lambda).$$

Let $\mathcal{P}(A)$ be the set of parabolic subgroups defined over $F$ and having $A$ as a Levi-factor. If $P$ is such a parabolic, there is $s$ such that $sP = B$. We set

$$c_P(\lambda) = c_s(\lambda), \quad d_P(\lambda) = d_s(\lambda). \tag{47}$$

We shall need to verify the following lemma.

LEMMA 11: *The families $(c_P)$ and $(d_P)$ are $(G, A)$ families in the sense of Arthur.*

*Proof:* This amounts to saying the following: suppose $s$ is a simple reflection and $s_1, s_2$ are such that $ss_1 = s_2$; if $\lambda$ verifies $s_2\lambda = s_1\lambda$ then

$$c_{s_2}(\lambda) = c_{s_1}(\lambda), \quad d_{s_2}(\lambda) = d_{s_1}(\lambda).$$

This is clear for the first family. For the second, the relation to be established reads: for $s_2\lambda = s_1\lambda$,

$$J(1, M(s_1,\lambda)I_\lambda(f_1^2)\phi', 0)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)} = J(1, M(s_2,\lambda)I_\lambda(f_1^2)\phi', 0)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}.$$

By linearity this reduces to

$$J(1, M(s_1,\lambda)\phi', 0)\overline{\mathcal{W}_\lambda(\phi)} = J(1, M(s_2,\lambda)\phi', 0)\overline{\mathcal{W}_\lambda(\phi)}.$$

Since $M(s_2,\lambda) = M(s, s_1\lambda)M(s_1,\lambda)$ we see that the relation we have to establish is

$$J(1, M(s,\lambda)\phi', 0)\overline{\mathcal{W}_\lambda(\phi)} = J(1, \phi', 0)\overline{\mathcal{W}_\lambda(\phi)}$$

if $s\lambda = \lambda$. Say $s = w_1$. There are two cases to consider.

Assume first that $\sigma_1 \neq \sigma_2$. Since the period is induced from $GL(2)$ it suffices to prove this relation in the context of $GL(2)$. That is, we consider the induced representation $I_{\lambda,\sigma_1,\sigma_2}$ and an Eisenstein series $E(g,\phi,\lambda;\sigma_1,\sigma_2)$. Then

$$\int \Lambda_m^T E(h,\phi,\lambda;\sigma_1,\sigma_2)dh = v_B \frac{e^{\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}\rangle}\int \phi(k)dk - v_B\frac{e^{-\langle\lambda,T\rangle}}{\langle\lambda,\check{\alpha}\rangle}\int M(\sigma,\lambda)\phi(k)dk.$$

The holomorphy at $\lambda = 0$ implies that, as claimed,

$$\int \phi(k)dk = \int M(\sigma,0)\phi(k)dk.$$

Now suppose that $\sigma_1 = \sigma_2$. The relation $w_1\lambda = \lambda$ is equivalent to $\lambda_0^Q = 0$. Then

$$M(w_2,\lambda) = M(s,\lambda_0^Q) = M(w_2, 0) = -1$$

(the last relation from the $GL(2)$ theory). Thus we cannot argue as before. However, $\mathcal{W}_\lambda$ vanishes for $\lambda_0^Q = 0$ which makes the above relation trivially true. Indeed, we have

$$\mathcal{W}_\lambda(I_\lambda(f)\phi) = \mathcal{W}_\lambda^1(I_\lambda(f)\phi)\frac{1}{\prod_{i<j} L^S(1 + s_i - s_j, \sigma_i\sigma_j^{-1})}$$

where $S$ is a suitable finite set of places. The linear form $\mathcal{W}_\lambda^1$ is defined locally and holomorphic for $\Re\lambda = 0$. Here $\sigma_1 = \sigma_2$. The relation $\lambda_0^Q = 0$ gives $s_1 = s_2$ so that the reciprocal of the $L$-factors vanishes as claimed. Our assertion follows. ∎

Now, with the notations of Arthur ([A3]),

$$J_\chi^{\min}(f) = \frac{n_1(\chi)}{n(\chi)}\sum_\phi \int \sum_{P\in\mathcal{P}(A)} c_P(\lambda)d_P(\lambda)\frac{1}{\theta_P(\lambda)}b(\lambda)d\lambda,$$

where we set $b(\lambda) = \langle I_\lambda(f_1^2)\phi, \phi'\rangle$. Following [A3], Lemma 6.3, we rewrite this as

$$\sum_{Q \supset A} c_A^Q(\lambda) d'_Q(\lambda).$$

We recall the definition of $c_A^Q$:

$$c_A^Q(\lambda) = \sum_{P \in \mathcal{P}(A), P \subseteq Q} c_P(\lambda)\theta_P^Q(\lambda)^{-1}.$$

Explicitly, for $Q \in \mathcal{P}(A)$ we have $Q = sB$ for some $s$ and then

$$c_A^Q(\lambda) = e^{\langle s\lambda, T\rangle}.$$

If $Q = G$ then

$$c_A^G(\lambda) = \sum_s \frac{e^{\langle s\lambda, T\rangle}}{\langle s\lambda, \check{\alpha}_1\rangle\langle s\lambda, \check{\alpha}_2\rangle}.$$

Suppose that $Q$ is a maximal proper parabolic subgroup containing $A$. Then $Q = sQ_i$ with $i = 1$ or $i = 2$. Here $Q_1$ (resp. $Q_2$) is the parabolic subgroup of type $(2,1)$ (resp. $(1,2)$). Then

$$c_A^Q(\lambda) = \frac{e^{\langle s\lambda, T\rangle}}{\langle s\lambda, \check{\alpha}_i\rangle} + \frac{e^{\langle s_i s\lambda, T\rangle}}{\langle s_i s\lambda, \check{\alpha}_i\rangle}.$$

We will not need the explicit value of the functions $d'_Q(\lambda)$ except for $d'_G(\lambda)$. In fact $d'_G(\lambda)$ is equal to the common value $d(0)$ of all the functions $d_P(\lambda)$, $P \in \mathcal{P}(A)$ at the point $\lambda = 0$. We will, however, need the fact that each function $d'_Q(\lambda)$ depends only on the projection $\lambda_Q$ on $i\mathfrak{A}_Q^*$ in the decomposition

$$\mathfrak{A}_0^* = \mathfrak{A}_0^{Q*} \oplus \mathfrak{A}_Q^*. \tag{48}$$

Each function $d'_Q(\lambda)$ is a smooth function on $i\mathfrak{A}_Q$ ([A3], Lemma 6.1). Moreover, it is of slow increase as well as all its derivatives. Indeed this follows from the proof of Lemma 6.1 in [A3] and the following elementary lemma:

LEMMA 12: *Suppose that $\phi$ is a smooth function, of slow increase as well as all its derivatives, on $\mathbb{R}^n$. If $\phi$ vanishes on the hyperplane $\{x | \langle x, \alpha\rangle = 0\}$, then*

$$\phi(x) = \langle x, \alpha\rangle \phi_0(x)$$

*where $\phi_0$ is a smooth function, of slow increase as well as all its derivatives.*

We recall the proof of the lemma. We can introduce a system of coordinates $(x_1, x_2, \ldots, x_n)$ such that the hyperplane is the coordinate hyperplane $x_1 = 0$.

Then we can take

$$\phi_0(x_1, x_2, \ldots, x_n) = \int_0^1 \frac{\partial \phi}{\partial x_1}(tx_1, x_2, \ldots, x_n).$$

Thus we can write our integral as

$$J_\chi^{\min}(f) = \frac{n_1(\chi)}{n(\chi)} v_Q \sum_{\phi,\phi'} \sum_{Q \supset A} \int c_A^Q(\lambda) d'_Q(\lambda) b(\lambda) d\lambda.$$

Recall that the factor $c_A^Q(\lambda)$ depends in fact on $T$.

LEMMA 13: *The following limit exists:*

$$\lim_{T\to\infty} \int c_A^Q(\lambda) d'_Q(\lambda) b(\lambda) d\lambda.$$

*If $Q = G$ and the $\sigma_i$ are distinct, then the limit is $d'_G(0)b(0)$. Otherwise, that is, if $Q \neq G$ or if the $\sigma_i$ are not all distinct, then the limit is zero.*

*Proof:* Let $\chi_T$ be the characteristic function of the convex hull of the points $\{s^{-1}T\}$ where $s$ runs over $W$. The function $c_A^G(\lambda)$ is the Fourier transform of $\chi_T$ ([A3], p. 36). Therefore

$$\begin{aligned}\lim_{T\to\infty} \int c_A^G(\lambda) d'_G(\lambda) b(\lambda) d\lambda &= \lim_{T\to\infty} \int \chi_Y(H) \widehat{d'_G b}(H) dH \\ &= \int \chi_Y(H) \widehat{d'_G b}(H) dH \\ &= d'_G(0) b(0).\end{aligned}$$

If the $\sigma_i$ are not distinct, then $d'_G(0) = 0$ since, as seen above, it involves the Whittaker function $\mathcal{W}_\lambda(I_\lambda(f_2)\phi)$ which vanishes at $\lambda = 0$.

Next, suppose that $Q = sQ_i$ is a maximal parabolic subgroup containing $A$. We may decompose $T$ as $T = T_{Q_i} + T_0^{Q_i}$ according to the direct sum in (48). Set $T_Q = s^{-1}T_{Q_i}$ and $T_0^Q = s^{-1}T_Q$, and let $\alpha = s^{-1}\alpha_i$. Then

$$c_A^Q(\lambda) = e^{\langle \lambda, T_Q\rangle} \Big( \frac{e^{\langle \lambda, T_0^Q\rangle}}{\langle \lambda, \check\alpha\rangle} - \frac{e^{-\langle \lambda, T_0^Q\rangle}}{\langle \lambda, \check\alpha\rangle} \Big),$$

and

$$\int c_A^Q(\lambda) d'_Q(\lambda) b(\lambda) d\lambda \tag{49}$$

can be written

$$\int_{\mathfrak{A}_Q^*} e^{\langle \lambda_Q, T_Q\rangle} d'_Q(\lambda_Q) \Big( \int_{\mathfrak{A}_0^{Q*}} \Big( \frac{e^{\langle \lambda, T_0^Q\rangle}}{\langle \lambda, \check\alpha\rangle} - \frac{e^{-\langle \lambda, T_0^Q\rangle}}{\langle \lambda, \check\alpha\rangle} \Big) b(\lambda_0^Q + \lambda_Q) d\lambda_0^Q \Big) d\lambda_Q.$$

The inner integral can be written in the form

$$\int \chi_{T_0^Q}(H)\chi_{T_0^Q}\hat{b}(H,\lambda_Q)dH,$$

where $\hat{b}(\bullet,\lambda_Q)$ is the Fourier transform of the function $\lambda_0^Q \mapsto b(\lambda_0^Q+\lambda_Q)$ and $\chi_{T_0^Q}$ is the characteristic function of an interval (depending on $T_0^Q$). The inner integral represents a Schwartz function of $\lambda_Q$, depending on the parameter $T_0^Q$. In particular, this Schwartz function remains in a fixed bounded set (independent of $T_0^Q$) of the space of Schwartz functions. Thus (49) is then the Fourier transform of this function evaluated at $T_Q$ and hence tends to zero as $T$ (and hence also $T_Q$) tends to infinity.

The Riemann–Lebesgue Lemma applies directly in the case $Q = sB$. ∎

Finally, we see that

$$\begin{aligned} J_\chi^{\min}(f) &= \frac{n_1(\chi)}{n(\chi)} v_B \sum_{\phi,\phi'} J(1, I_0(f_1^1)\phi', 0)\overline{\mathcal{W}_0(I_0(f_2)\phi)}\langle I_0(f_1^2)\phi, \phi'\rangle \\ &= v_B \sum_{\phi} J(1, I_0(f_1)\phi, 0)\overline{\mathcal{W}_0(I_0(f_2)\phi)}. \end{aligned}$$

As before this can be written in terms of generalized vectors. Note that $n_1(\chi) = n(\chi)$ if the $\sigma_i$ are distinct. Also, $J_\chi^{\max}(f) = 0$ if the $\sigma_i$ are distinguished and distinct.

Proposition 9: *Suppose that $\chi$ is represented by three characters $\sigma_i$, $1 \le i \le 3$. The term $J_\chi^{\min}(f)$ vanishes unless the characters are distinct and distinguished. If they are distinguished and distinct, then $J_\chi(f) = J_\chi^{\min}(f)$ and, for any smooth function of compact support $f$,*

$$J_\chi(f) = v_B \langle I_{0,\sigma}(f)\mathcal{W}_{0,\sigma}, J_{0,\sigma}\rangle. \tag{50}$$

This identity has been established when $f$ is a $K$-finite function which is itself a sum of sufficiently many convolution products of $K$-finite functions. Since both sides are distributions, the equality is in fact true for all $f$.

8.2 The term $J_\chi^{\max}(f)$. This terms vanishes unless the following condition is satisfied:

(∗) One of the three characters, $\sigma_i$ say, is distinguished and the two others $\sigma_j$ and $\sigma_k$ are related by $\sigma_j(a) = \sigma_k(\overline{a})^{-1}$.

If either $\sigma_j$ or $\sigma_k$ is distinguished, then $\sigma_j = \sigma_k$.

According to Proposition 9, $J_\chi^{\min}(f) = 0$ unless the $\sigma_j$ are distinguished and distinct. It follows that

$$J_\chi(f) = J_\chi^{\max}(f)$$

when (*) is satisfied. Without loss of generality, we may assume that

$$\sigma_1(a) = \sigma_2(\overline{a})^{-1}, \qquad \sigma_3 \text{ is distinguished.}$$

Let $m(\chi) = \frac{1}{2}$ if $\sigma_1 = \sigma_2$ and $m(\chi) = 1$ if $\sigma_1 \neq \sigma_2$.

PROPOSITION 10: *Suppose $\chi$ is represented by $(\sigma_1, \sigma_2, \sigma_3)$ where $\sigma_1(a) = \sigma_2(\overline{a})^{-1}$ and $\sigma_3$ is distinguished. Then $J_\chi(f) = J_\chi^{\max}(f)$ and, for all smooth functions of compact support,*

$$J_\chi(f) = m(\chi) v_Q \int \langle I_{\lambda,\sigma}(f)\mathcal{W}_{\lambda,\sigma}, J_{\lambda,\sigma}\rangle d\lambda,$$

*where $\mathcal{W}_{\lambda,\sigma}$ is as before and $J_{\lambda,\sigma}$ is the generalized vector defined below. Moreover, if $\mathcal{B}$ is a bounded set, there is $C > 0$ such that for $f \in \mathcal{B}$,*

$$\sum_\chi m(\chi) \int |\langle I_{\lambda,\sigma}(f)\mathcal{W}_{\lambda,\sigma}, J_{\lambda,\sigma}\rangle|\, d\lambda < C,$$

*the sum over all $\chi$ of the above type.*

To begin the proof, we note that

$$\text{(51)} \qquad \mathrm{I}(Q, \phi, \lambda) = \sum_w v_Q \frac{e^{\langle (w\lambda)_Q, T\rangle}}{\langle (w\lambda)_Q, \check{\alpha}_Q\rangle} J(\xi_Q, M(w,\lambda)\phi, (w\lambda)_0^Q)$$

and

$$\text{(52)} \qquad \mathrm{I}(Q', \phi, \lambda) = \sum_w v_{Q'} \frac{e^{\langle (w\lambda)_{Q'}, T\rangle}}{\langle (w\lambda)_{Q'}, \check{\alpha}_{Q'}\rangle} J(\xi_{Q'}, M(w,\lambda)\phi, (w\lambda)_0^{Q'}).$$

In (51), the sum is over all $w$ such that $w^{-1}\alpha_1 > 0$ and $w\sigma$ is trivial on the torus $T_{\eta_Q}$, that is, the torus of matrices of the form

$$\mathrm{diag}(\alpha, \overline{\alpha}, \beta), \quad \alpha \in E, \beta \in F.$$

Likewise in (52), the sum is over all $w$ such that $w^{-1}\alpha_2 > 0$ and $w\sigma$ is trivial on the torus $T_{\eta_{Q'}}$.

We shall need to use the following functional equation for the intertwining period.

LEMMA 14: *With the previous notations, if* $\lambda = \lambda_0^Q$ *is in* $i\mathfrak{a}_0^{Q*}$, *then*

$$J(\xi_Q, \phi, \lambda) = J(\xi_{Q'}, M(w_1 w_2, \lambda)\phi, w_1 w_2 \lambda).$$

*Proof:* By Proposition 33 of [JLR]

$$J(\xi_Q, \phi, \lambda) = J(w_1 \xi_Q w_1, M(w_2, \lambda)\phi, w_2\lambda).$$

Similarly, if $\lambda' \in i\mathfrak{a}_0^{Q'*}$ then

$$J(\xi_{Q'}, \phi, \lambda') = J(w_1 \xi_{Q'} w_1, M(w_1, \lambda')\phi', w_1\lambda').$$

Now

$$w_1 \xi_Q w_1 = w_2 \xi_Q w_2.$$

Moreover, if $\lambda' = w_1 w_2 \lambda$ and $\phi' = M(w_1 w_2, \lambda)\phi$ then

$$w_2\lambda = w_1\lambda' \quad \text{and} \quad M(w_2, \lambda)\phi = M(w_1, \lambda')\phi'.$$

The lemma follows. ∎

*Remark:* If $\sigma_1$ and $\sigma_2$ are not distinguished, then we can derive the functional equation from the formula

$$(53) \quad \int \Lambda_m^T E(h, \phi, \lambda) dh = v_Q \frac{e^{\langle \lambda_Q, T\rangle}}{\langle \lambda_Q, \check{\alpha}_Q\rangle} J(\xi_Q, \phi, \lambda_0^Q) - v_Q \frac{e^{-\langle \lambda_Q, T\rangle}}{\langle \lambda_Q, \check{\alpha}_Q\rangle} J(\xi_{Q'}, M(w_1 w_2, \lambda_0^Q + \lambda_Q)\phi, w_1 w_2(\lambda_0^Q)).$$

Using the fact that the singularities on the line $\lambda_Q = 0$ cancel, we find

$$(54) \qquad J(\xi_Q, \phi, \lambda_0^Q) = J(\xi_{Q'}, M(w_1 w_2, \lambda_0^Q)\phi, w_1 w_2 \lambda_0^Q).$$

We also note that $J(\xi_Q, \phi, \lambda_0^Q)$ is induced from $GL(2)$ in the following sense:

$$J(\xi_Q, \phi, \lambda_0^Q) = \int_{K_H} \int_{T_Q(F_{\mathbb{A}})\backslash M_H(F_{\mathbb{A}})} e^{\langle \lambda_0^Q + \rho_Q, H(\eta_Q m)\rangle} \phi(\eta_Q m k) dm dk$$

where $M_H$ is the Levi subgroup of type $(2,1)$ in $H$. Now $M_H$ is isomorphic to $GL(2,F)$ times $GL(1,F)$. Thus the integral over $M_H$ is really an integral defining the intertwining period for the representation $I(\sigma_1, \sigma_2)$ of $GL(2)$, or more precisely, its analytic continuation (section 5.3). It inherits the analytic properties of the intertwining period for $GL(2)$. In particular, the following result follows directly from the integral representation (31).

LEMMA 15: *If $\sigma_1(a) = \sigma_2(\overline{a})^{-1}$ but $\sigma_1$ is not distinguished, then*

$$J(\xi_Q, \phi, \lambda_0^Q)$$

*has no singularity on the set $\Re\lambda_O^Q = 0$. If $\sigma_1(a) = \sigma_2(\overline{a})^{-1}$ and $\sigma_1$ is distinguished, then $J(\xi_Q, \phi, \lambda_0^Q)$ has a simple pole at $\lambda_0^Q = 0$.*

More precisely, we can use the $GL(2)$ theory to define, for $S$ large enough, a normalized intertwining period $J^1$ such that

$$J(\xi_Q, I_\lambda(f_1^1)\phi, \lambda_0^Q) = J^1(\xi_Q, I_\lambda(f_1^1)\phi, \lambda_0^Q)\frac{L^{S_0}(1-2s, \sigma_1^{-1}|F)}{L^{S_0}(1+2s, \omega_{E/F}\sigma_1|F)}. \tag{55}$$

The normalized period $J^1$ is defined locally and has no singularity on the line $\Re\lambda = 0$.

The corresponding relation for $\mathcal{W}$ is

$$\begin{aligned}\mathcal{W}_\lambda(I_\lambda(f)\phi) = \mathcal{W}_\lambda^1(I_\lambda(f)\phi) &\times \frac{1}{L^{S_0}(1+2s, \sigma_1|F)L^{S_0}(1+2s, \omega_{E/F}\sigma_1|F)} \\ &\times \frac{1}{\prod_{i<j,(i,j)\neq(1,2)} L^S(1+s_i-s_j, \sigma_i\sigma_j^{-1})}.\end{aligned} \tag{56}$$

Suppose first that $\sigma_1 = \sigma_2 = \sigma_3$. Then the condition that $w\sigma$ be trivial on $T_{\eta_Q}$ or $T_{\eta_{Q'}}$ is vacuous. Thus in (51) the sum is over $w \in \{e, w_2, w_2w_1\}$ and, in (52), the sum is over $w \in \{e, w_1, w_1w_2\}$. In this case $n_1(\chi)/n(\chi) = 1/6$. It will be convenient to introduce the following expressions:

$$\begin{aligned}A_1(\lambda,\phi) =& v_Q\frac{e^{\langle\lambda_Q, T\rangle}}{\langle\lambda_Q, \check{\alpha}_Q\rangle}J(\xi_Q, \phi, \lambda_0^Q) \\ &+ v_{Q'}\frac{e^{\langle(w_1w_2\lambda)_{Q'}, T\rangle}}{\langle(w_1w_2\lambda)_{Q'}, \check{\alpha}_{Q'}\rangle}J(\xi_{Q'}, M(w_1w_2, \lambda)\phi, (w_1w_2\lambda)_0^Q), \\ A_2(\lambda,\phi) =& v_{Q'}\frac{e^{\langle\lambda_{Q'}, T\rangle}}{\langle\lambda_{Q'}, \check{\alpha}_{Q'}\rangle}J(\xi_{Q'}, \phi, \lambda_0^{Q'}) \\ &+ v_Q\frac{e^{\langle(w_2w_1\lambda)_Q, T\rangle}}{\langle(w_2w_1\lambda)_Q, \check{\alpha}_Q\rangle}J(\xi_Q, M(w_2w_1, \lambda)\phi, (w_2w_1\lambda)_0^Q), \\ A_3(\lambda,\phi) =& v_Q\frac{e^{w_2\langle\lambda_Q, T\rangle}}{\langle(w_2\lambda)_Q, \check{\alpha}_Q\rangle}J(\xi_Q, M(w_2, \lambda)\phi, (w_2\lambda)_0^Q) \\ &+ v_{Q'}\frac{e^{\langle(w_1\lambda)_{Q'}, T\rangle}}{\langle(w_1\lambda)_{Q'}, \check{\alpha}_{Q'}\rangle}J(\xi_{Q'}, M(w_1, \lambda)\phi, (w_1\lambda)_0^Q).\end{aligned}$$

The following relations are easily checked:

$$A_1(w_2w_1\lambda, M(w_2w_1, \lambda)\phi) = A_2(\lambda, \phi), \quad A_1(w_2\lambda, M(w_2, \lambda)\phi) = A_3(\lambda, \phi).$$

The functional equation of the Eisenstein series implies that

$$\mathcal{W}_{w\lambda}(M(w,\lambda)\phi) = \mathcal{W}_\lambda(\phi).$$

Thus altogether $J_\chi^{\max}(f)$ is 1/6 times the integral of the sum of the following terms:

$$A(\lambda) = \sum_\phi A_1(\lambda,\phi)\overline{\mathcal{W}_\lambda(I_\lambda f_2)\phi)},$$

$$B(\lambda) = \sum_\phi A_1(w_2w_1\lambda, M(w_2w_1,\lambda)\phi)\overline{\mathcal{W}_{w_2w_1\lambda}(I_{w_2w_1\lambda}(f_2)M(w_2w_1,\lambda)\phi)},$$

$$C(\lambda) = \sum_\phi A_1(w_2\lambda,\lambda,\phi)\overline{\mathcal{W}_{w_2\lambda}(I_{w_2\lambda}(f_2)(M(w_2,\lambda)\phi)}.$$

Since the sums over the orthonormal basis are independent of the choice of the basis, we get in fact

$$B(\lambda) = A(w_2w_1\lambda), \quad C(\lambda) = A(w_2\lambda).$$

We note the simplifying relations

$$v_{Q'} = v_Q,$$
$$\langle (w\lambda)_{Q'}, T\rangle = -\langle \lambda_Q, T\rangle, \quad \langle (w\lambda)_{Q'}, \check{\alpha}_{Q'}\rangle = -\langle \lambda_Q, \check{\alpha}_Q\rangle,$$
$$(w\lambda)_0^{Q'} = w(\lambda_0^Q).$$

We will show in a moment that $A(\lambda)$ is a smooth integrable function of $\lambda$. After a change of variables, it will follow that $J_\chi^{\max}(f)$ is equal to the integral of $\frac{1}{2}A(\lambda)$.

If $\sigma_1 \neq \sigma_3$, then also $\sigma_2 \neq \sigma_3$ and the sum in (51) reduces to a single term, $w = e$. The sum in (52) reduces to a single term, $w = w_1w_2$. Again we will show that $A(\lambda)$ is a smooth integrable function of $\lambda$. Furthermore, $n_1(\chi)/n(\chi) = 1/2$. Defining $A(\lambda)$ as before, we will show that $A(\lambda)$ is a smooth integrable function of $\lambda$. It will follow that $J_\chi^{\max}(f)$ is also equal to the integral of $\frac{1}{2}A(\lambda)$ in this case.

Set

$$d_{+,\phi,\phi'}(\lambda) = J(\xi_Q, I_\lambda(f_1^1)\phi', \lambda_0^Q)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}, \tag{57}$$

$$d_{-,\phi,\phi'}(\lambda) = J(\xi_{Q'}, M(w,\lambda)I_\lambda(f_1^1)\phi', w\lambda_0^Q)\overline{\mathcal{W}_\lambda(I_\lambda(f_2)\phi)}. \tag{58}$$

Then, in all cases where (*) is satisfied, $A(\lambda)$ is equal to

$$\sum_{\phi,\phi'}\left[\frac{e^{\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check{\alpha}_Q\rangle}d_{+,\phi,\phi'}(\lambda) - \frac{e^{-\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check{\alpha}_Q\rangle}d_{-,\phi,\phi'}(\lambda)\right]\langle I_\lambda(f_1^2)\phi,\phi'\rangle.$$

LEMMA 16: *The functions*

$$d_{+,\phi,\phi'}, d_{-,\phi,\phi'}$$

*have no singularities on* $\Re\lambda = 0$. *On* $\Re\lambda = 0$ *they are of slow increase as well as all their derivatives. Furthermore, the relation*

$$d_{+,\phi,\phi'}(\lambda_0^Q) = d_{-,\phi,\phi'}(\lambda_0^Q)$$

*holds. In particular,* $A(\lambda)$ *is a smooth integrable function of* $\lambda$.

*Proof:* Lemma (14) implies the relation of the lemma. Then the assertion on $A(\lambda)$ follows from the first part of the Lemma.

We check the assertions relative to $d_{+,\phi,\phi'}$. Indeed,

$$\begin{aligned} d_{+,\phi,\phi'}(\lambda) =& J^1(\xi_Q, I_\lambda(f_1^1)\phi', \lambda_0^Q)\overline{W_\lambda^1(I_\lambda(f_2)\phi)} \\ &\times \frac{1}{\left|L^{S_0}(1+2s, \omega_{E/F}\sigma_1|F)\right|^2} \\ &\times \frac{1}{\prod_{i<j,(i,j)\neq(1,2)} L^S(1+s_j-s_i, \sigma_j\sigma_i^{-1})}. \end{aligned}$$

This expression has no singularity on $\Re\lambda = 0$ and is of slow increase as well as its derivatives.

For $d_{-,\phi,\phi'}$ the proof is similar, but there is an extra step because of the presence of the intertwining operator. We have to assume that $f_1^1$ is itself a convolution product $f_1^1 = f_1^3 * f_1^4$ and then write

$$J(\xi_{Q'}, \dots) = \sum_{\phi''} J(\xi_{Q'}, I_{w\lambda}(f_1^3)\phi'', w\lambda_0^Q)\langle M(w,\lambda)I_\lambda(f_1^4)\phi', \phi''\rangle.$$

The factor $\langle M(w,\lambda)I_\lambda(f_1^4)\phi', \phi''\rangle$ has derivatives of slow increase so we can finish the proof as before.

It now follows that $J_\chi^{\max}(f)$ is equal to

$$m(\chi)v_Q\sum_{\phi,\phi'}\int\left[\frac{e^{\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check{\alpha}_Q\rangle}d_{+,\phi,\phi'}(\lambda) - \frac{e^{-\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check{\alpha}_Q\rangle}d_{-,\phi,\phi'}(\lambda)\right]\langle I_\lambda(f_1^2)\phi,\phi'\rangle d\lambda$$

(note that $v_Q = v_{Q'}$). We shall compute the limit as $T \to \infty$ by the method of the $(G, M)$ families, which amounts here to writing the integrand as the sum of the following three terms:

$$e^{\langle\lambda_Q,T\rangle}\frac{d_{+,\phi,\phi'}(\lambda) - d_{+,\phi,\phi'}(\lambda_0^Q)}{\langle\lambda_Q,\check{\alpha}_Q\rangle}\langle I_\lambda(f_1^2)\phi,\phi'\rangle,$$

$$-e^{-\langle\lambda_Q,T\rangle}\frac{d_{-,\phi,\phi'}(\lambda)-d_{+,\phi,\phi'}(\lambda_0^Q)}{\langle\lambda_Q,\check\alpha_Q\rangle}\langle I_\lambda(f_1^2)\phi,\phi'\rangle,$$

$$\Big(\frac{e^{\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check\alpha_Q\rangle}-\frac{e^{-\langle\lambda_Q,T\rangle}}{\langle\lambda_Q,\check\alpha_Q\rangle}\Big)d_{+,\phi,\phi'}(\lambda_0^Q)\langle I_\lambda(f_1^2)\phi,\phi'\rangle.$$

From the relation

$$d_{+,\phi,\phi'}(\lambda_0^Q)=d_{-,\phi,\phi'}(\lambda_0^Q)$$

we see that in the two first terms the fractions represent smooth functions of $\lambda$ which are of slow increase. Since $\langle I_\lambda(f_1^2)\phi,\phi'\rangle$ is of rapid decrease, the product is integrable and the integrals of the first two terms tend to 0 as $\to\infty$ by the Riemann–Lebesgue Lemma. In the third term, the expression in parentheses is the Fourier transform of the convex hull of $-T_Q, T_Q$ where $T_Q$ is the projection of $T$ on $\mathfrak{a}_Q$. Hence its integral tends to

$$\int_{i\mathfrak{A}_0^Q} d_{+,\phi,\phi'}(\lambda_0^Q)\langle I_{\lambda_0^Q}(f_1^2)\phi,\phi'\rangle d\lambda_0^Q.$$

Thus we finally find

$$J_\chi^{\max}(f)=m(\chi)v_Q\sum_\phi\int_{i\mathfrak{A}_0^{Q*}} J(\xi_Q,I_{\lambda_0^Q}(f_1)\phi,\lambda_0^Q)\overline{W}_\lambda(I_{\lambda_0^Q}(f_2)\phi,\lambda_0^Q)d\lambda_0^Q.$$

This formula has been established for a $K$-finite function $f=f_1*f_2^*$ where $f_1$ and $f_2$ are $K$-finite and themselves convolution of $K$-finite functions. We will write $\lambda$ for $\lambda_0^Q$ and set $s=\langle\lambda,\check\alpha_1\rangle$. We will also indicate the dependence on $\sigma$. We have seen that $\mathcal{W}_{\lambda,\sigma}$ and $J(\xi_Q,\phi,\lambda,\sigma)$ (which may not be defined at $\lambda=0$) define continuous linear forms on the space of smooth vectors of the representation $I_{\lambda,\sigma}$. Denoting by $\mathcal{W}_{\lambda,\sigma}$ and $J_{\lambda,\sigma}$ the corresponding generalized vectors, we can write the result in the form

$$J_\chi^{\max}(f)=m(\chi)v_Q\int\langle I_{\lambda,\sigma}(f)\mathcal{W}_{\lambda,\sigma},J_{\lambda,\sigma}\rangle d\lambda.$$

The integrand is defined even at $\lambda=0$. In fact, we can write the integrand in the form

$$\begin{aligned}&\langle I_{\lambda,\sigma}(f)\mathcal{W}^1_{\lambda,\sigma},J^1_{\lambda,\sigma}\rangle\\ &\times\frac{1}{\left|L^{S_0}(1+2s,\sigma_1|F\omega_{E/F})\right|^2}\\ &\times\frac{1}{\prod_{i<j,(i,j)\neq(1,2)}L^S(1+s_j-s_i,\sigma_j\sigma_i^{-1})}.\end{aligned}$$

For our next step we let $S_0$ be a finite set of places of $F$ and $S$ the corresponding set of places of $E$. As usual we assume that $S_0$ contains all places at infinity. For $v$ finite we fix a smooth function of compact support $f_v$. We assume it is the characteristic function of $K_v$ for almost all $v$. For $v$ infinite we let $\mathcal{B}_v$ be a bounded set of the space of smooth functions of compact support. Finally we let $\mathcal{B}$ be the set of functions $f = \prod f_v$ with $f_v \in \mathcal{B}_v$ for $v$ infinite.

LEMMA 17: *Let $\mathcal{B}$ be the above set. Then there is a constant $C > 0$ such that, for $f \in \mathcal{B}$,*

$$\sum_\sigma \int |\langle I_{\lambda,\sigma}(f)\mathcal{W}_{\lambda,\sigma}, J_{\lambda,\sigma}\rangle| \, d\lambda \leq C,$$

*the sum over all triples of the above type.*

To prove the lemma, we may as well assume that $f = f_1 * f_2 * f_3$ where the functions $f_i$, $i = 1, 2, 3$ remain in a set $\mathcal{B}$. Using local majorizations, we see that there is a polynomial $P(\lambda, \sigma)$ such that for $\phi$ of norm one and $f \in \mathcal{B}$,

$$\left|J^1_{\lambda,\sigma}(I_{\lambda,\sigma}(f)\phi)\right| \leq |P(s, \sigma)|.$$

Otherwise,

$$\| I_{\lambda,\sigma}(f) J^1_{\lambda,\sigma} \| \leq |P(s, \sigma)|.$$

There is a similar estimate for $\mathcal{W}^1_{\lambda,\sigma}$. We can write

$$\langle I_{\lambda,\sigma}(f)\mathcal{W}_{\lambda,\sigma}, J_{\lambda,\sigma}\rangle = \langle I_{\lambda,\sigma}(f_2) I_{\lambda,\sigma}(f_3)\mathcal{W}^1_{\lambda,\sigma}, I_{\lambda,\sigma}(f_1^*) J^1_{\lambda,\sigma}\rangle$$

and this is thus majorized by

$$\| I_{\lambda,\sigma}(f_2) \| \, |P^1(s, \sigma)|,$$

where $P^1$ is a polynomial. Using estimates for the $L$-factors, we are reduced to estimate an expression of the form

$$\sum_\sigma \int \| I_{\lambda,\sigma}(f) \| \, |P(s, \sigma)| \, d\lambda,$$

for $f \in \mathcal{B}$. Now $I_{\lambda,\sigma}(f)$ is the operator defined by the kernel

$$L(k_1, k_2) = \int f(k_1^{-1} a n k_2) dn e^{\langle \lambda + \rho, H(a)\rangle} dn da.$$

Thus

$$\| I_{\lambda,\sigma}(f_2) \| \leq \sup_{k_1, k_2} |L(k_1, k_2)|.$$

For $k_i \in K$, $f \in \mathcal{B}$ the functions

$$a \mapsto \int f(k_1^{-1} a n k_2) dn e^{\langle \rho, H(a) \rangle}$$

remain in a bounded set $\mathcal{B}_A$ of the space of smooth functions of compact support on $A(\mathbb{A})$. In particular, they are invariant under $A(E_{\mathbb{A}}) \cap K^S$. We have then

$$\| I_{\lambda,\sigma}(f_2) \| \leq \sup_{\phi \in \mathcal{B}_A} \left| \int \phi(a) \sigma(a) e^{\langle \lambda, H(a) \rangle} da \right| .$$

Finally, given $P(\lambda, \sigma)$ and a bounded set $\mathcal{B}_A$, we need to show that there is a constant $C$ such that

$$\sum_{\sigma} \int \left| \int \phi(a) \sigma(a) e^{\langle \lambda, H(a) \rangle} da \right| |P(\lambda, \sigma)| d\lambda \leq C,$$

for $\phi \in \mathcal{B}_A$. Our assertion follows.

Now we finish the proof of Proposition 10. By the previous lemma,

$$f \mapsto m(\chi) v_Q \int \langle I_{\lambda,\sigma}(f) \mathcal{W}_{\lambda,\sigma}, J_{\lambda,\sigma} \rangle d\lambda$$

is a distribution. It is equal to $J_\chi(f)$ when $f$ is $K$-finite and a convolution of sufficiently many $K$-finite functions. Thus it is in fact equal to $J_\chi(f)$ for all $f$.

## 9. The discrete part of the trace formula

Now we claim that, for any smooth function of compact support,

$$J(f) = \sum_{\text{discrete}} J_\chi(f) + \sum_{\mu_1} J_{\mu_1}(f). \tag{59}$$

In the first sum, $\chi$ appears if it is represented by a pair $(M, \pi)$ where $\pi$ is a distinguished representation of $M$ and, furthermore, $n(\chi) = n_1(\chi)$. The restriction of the central character of $\pi$ to center of $G$ must be equal to $\omega$. Then

$$J_\chi(f) = \langle I_\pi(f) \mathcal{W}_\pi, J_\pi \rangle,$$

where $\mathcal{W}_\pi$ is the generalized vector corresponding to a Whittaker linear form and $J_\pi$ the generalized vector corresponding to the period integral

$$J_\pi(\phi) = \int_{K_H} \int_{M_H} \phi(hk) dh dk.$$

Explicitly, the possibilities are $M = G$, $M$ is of type $(2,1)$ and $M = A$, with $\pi = (\sigma_1, \sigma_2, \sigma_3)$, the three characters being distinct and distinguished. The second sum is over all idele class characters $\mu_1$ of $E$. Then

$$J_{\mu_1}(f) = \frac{1}{2} v_Q \int \langle I_{\lambda,\pi}(f) \mathcal{W}_{\lambda,\pi}, J_{\lambda,\pi} \rangle d\lambda$$

where $\pi$ is the representation induced by $(\mu_1, \overline{\mu}_1^{-1}, \mu_2)$, $\mu_2$ is the distinguished idele class character of $E$ such that $\overline{\mu}_1^{-1}\mu_2 = \omega$, and $\lambda$ is integrated over the dual of $i\mathfrak{a}_0^Q$. Furthermore, the expression is absolutely convergent. More precisely, if $\mathcal{B}$ is a bounded set in the space of smooth functions of compact support on $G(E_{\mathbb{A}})$ then there is $C > 0$ such that, for $f \in \mathcal{B}$,

$$\sum_{\text{discrete}} |J_\chi(f)| + \sum \frac{1}{2} v_Q \int |\langle I_{\lambda,\pi}(f) \mathcal{W}_{\lambda,\pi}, J_{\lambda,\pi} \rangle| \, d\lambda < C.$$

To prove the equality (59), we remark that we have established it when $f$ is the convolution of sufficiently many $K$-finite functions. Since both sides are distributions, they must be equal for all functions $f$.

## 10. Convergence for the unitary group

We go back to the notations of the introduction. Thus $U$ is the unitary group for the matrix $w$. If $\mu$ is an idele class character of $E$, we consider the space $I(\mu, \zeta)$ of smooth functions $\phi: U(F_{\mathbb{A}}) \to \mathbb{C}$ such that

$$\phi(u d_a n g) = \phi(g) \mu(a) \zeta(u).$$

We set

$$e^{\langle \lambda + \rho, H(g) \rangle} = | \, a \, |^{1+s}$$

if $g = n' d_a u k$, with $k$ in the standard maximal compact subgroup $K'$. For $\phi$ a $K'$-finite function in $I(\mu, \sigma)$, we define the corresponding Eisenstein series

$$E(g, \phi; \lambda, \mu, \zeta) = \sum \phi(\gamma g) e^{\langle \lambda + \rho, H(\gamma g) \rangle}.$$

We define a linear form $\mathcal{W}_{\lambda,\mu,\zeta}$ on the space of $K$-finite vectors of $I(\mu, \zeta)$ by

$$\mathcal{W}_{\lambda,\mu,\zeta}(f) = \int E(n, \phi; \lambda, \mu, \zeta) \theta(n) dn.$$

This is the analytic continuation of the integral

$$\int \phi(wn) e^{\langle \lambda + \rho, H(wn) \rangle} \theta(n) dn.$$

Again, the linear form extends to the space of smooth vectors. In fact, if $S_0$ and $S$ are as before, then the restriction of the linear form to the space of vectors invariant under $K^{S_0}$ is given by

$$\mathcal{W}_{\lambda,\mu,\zeta}(\phi) = \mathcal{W}^1_{\lambda,\mu,\zeta}(\phi) \times \frac{1}{L^S(s+1,\mu)L^{S_0}(s+1,\omega_{E/F}\mu|F)},$$

where $\mathcal{W}^1_{\lambda,\mu,\zeta}$ is defined locally and holomorphic in $\lambda$. Let us now consider a datum $\chi$. Then the kernel associated to a $K$-finite function is given by

$$K_\chi(x,y) = \int \sum_\phi E(x, I_{\lambda,\mu,\chi}(f)\phi;\lambda)\overline{E(y,\phi;\lambda)}d\lambda + \cdots,$$

where the terms $\cdots$ correspond to residues of the Eisenstein series. For the purpose of showing that the trace formula gives an absolutely convergent expression, it is more convenient to define

$$J_\chi(f) = \int K_\chi(n_1,n_2)\theta(n_1)\overline{\theta(n_2)}dn_1dn_2.$$

Because the linear form $\mathcal{W}_{\lambda,\mu,\zeta}$ is holomorphic for $\Re s \geq 0$ and the residual spectrum is obtained by taking the residue at appropriate points for $\Re s > 0$, we get

$$J_\chi(f) = \int \sum_\phi \mathcal{W}_{\lambda,\mu,\zeta}(I_{\lambda,\mu,\zeta}(f)\phi)\overline{\mathcal{W}_{\lambda,\mu,\zeta}(\phi)}d\lambda.$$

We can introduce a generalized vector $\mathcal{W}_{\lambda,\mu,\zeta}$ and write the above expression as

$$J_\chi(f) = \int \langle I_{\lambda,\mu,\zeta}(f)\mathcal{W}_{\lambda,\mu,\zeta}, \mathcal{W}_{\lambda,\mu,\zeta}\rangle d\lambda.$$

We want to prove the following: if $f$ is a smooth function of compact support, then

$$\sum_\mu \int |\langle I_{\lambda,\mu,\zeta}(f)\mathcal{W}_{\lambda,\mu,\zeta}, \mathcal{W}_{\lambda,\mu,\zeta}\rangle|\, d\lambda < +\infty.$$

We may assume that $f$ is a convolution product and even that

$$f = f_1^* * f_1,$$

where $f_1$ is a smooth function of compact support. Then the expression reads

$$\sum_{\mu,\zeta} \int ||I_{\lambda,\mu,\zeta}(f_1)\mathcal{W}_{\lambda,\mu,\zeta}||^2 d\lambda.$$

If $f_1$ is $K$-finite, this is indeed finite. In fact, there is a continuous semi-norm $m$ on the space of smooth functions of compact support such that the above expression is bounded by $m(f_1^* * f_1)$ for $f_1$ $K$-finite. The general case can be obtained in the following way: given $f_1$ we may write $f_1 = \lim f_\alpha$ where $(f_\alpha)$ is a sequence of smooth $K$-finite functions of compact support. The functions $f_1$ and $(f_\alpha)$ have support in a fixed compact set and $f_\alpha$ converges uniformly to $f$ and any invariant derivative of $f_\alpha$ converges uniformly to the corresponding derivative of $f_1$. We have

$$\sum_\mu \int \|I_{\lambda,\mu,\zeta}(f_\alpha - f_\beta)\mathcal{W}_{\lambda,\mu,\zeta}\|^2 d\lambda \leq m((f_\alpha - f_\beta)(f_\alpha - f_\beta)^*).$$

As $\alpha$ and $\beta$ tend to infinity the right hand side tends to 0. We may regard all the Hilbert spaces $I_\mu$ as closed subspaces of the same Hilbert space $\mathcal{H} := L^2(K, dk)$. We may also regard $X = i\mathfrak{A}_0^* \times \widehat{E_\mathbb{A}^\times \backslash E^\times}$ as a measure space $(X, dx)$. The functions

$$(\lambda, \mu\zeta) \mapsto I_{\lambda,\mu,\zeta}(f)\mathcal{W}_{\lambda,\mu,\zeta}$$

are then measurable functions on $X$ with values in $\mathcal{H}$. Then the sequence of functions

$$A_\alpha \colon (\lambda, \mu) \mapsto I_{\lambda,\mu,\zeta}(f_\alpha)\mathcal{W}_{\lambda,\mu,\zeta}$$

is a Cauchy sequence in the Hilbert space $L^2(X, dx, \mathcal{H})$. It converges pointwise to the function

$$A_1 \colon (\lambda, \mu) \mapsto I_{\lambda,\mu,\zeta}(f_1)\mathcal{W}_{\lambda,\mu,\zeta}.$$

Thus a subsequence of $A_\alpha$ converges pointwise and in the Hilbert space to the function $A_1$. It follows that the function $A_1$ is in the same Hilbert space. Moreover,

$$\sum_\mu \int \|I_{\lambda,\mu,\zeta}(f_1)\mathcal{W}_{\lambda,\mu,\zeta}\|^2 d\lambda \leq m(f_1^* * f_1).$$

## 11. Conclusion

We are now ready to prove the main result of this paper.

THEOREM 2: *Any stable tempered packet of cuspidal representations of $U$ contains a globally generic representation.*

*Proof:* Indeed the packet has for base change a single irreducible cuspidal representation $\Pi$ of $GL(3, E_\mathbb{A})$. By the results discussed in the second section the representation $\Pi$ is distinguished. We go back to the discussion of the introduction section. If $f$ and $f'$ are as in the introduction, the absolute convergence of the relative trace formula allows us to write

$$J_\Pi(f) = \sum_\pi J'_\pi(f')$$

where the sum is over all the members of the packet. Since the distribution $J_\Pi(f)$ is a product of local relative Bessel distributions, we can choose a function $f$ satisfying our simplifying assumption such that the left hand side is non-zero. It follows that the right hand side is non-zero and there is at least one $\pi$ such that $J'_\pi(f') \neq 0$. Such a $\pi$ is generic. ∎

The same proof applies to the case of an endoscopic packet, by considering the discrete, non-cuspidal, part of the trace formula. At any rate, the result is already known as a consequence of the theory of the Weil representation ([GRS]). It is also a consequence of another trace formula (cf. [F4], and also [Mao2]).

We can also prove a local result.

THEOREM 3: *Let $E/F$ be a local quadratic extension. Any tempered $L$-packet of $U(F)$ contains exactly one generic component.*

*Proof:* By [GRS2], the only case which is not known is the case of a local packet reduced to a single supercuspidal representation $\pi$. Let us write the local quadratic extension as $E_v/F_{v_0}$ where $E/F$ is a quadratic extension of number fields. Let us write $\pi_{v_0}$ for $\pi$. There is an irreducible cuspidal representation $\pi$ of $U(F_\mathbb{A})$ having the component $\pi_{v_0}$. By the previous theorem there is a cuspidal generic representation $\pi'$ in the same $L$-packet as $\pi$. In particular $\pi'_{v_0} = \pi_{v_0}$, and the theorem follows. ∎

We have also the following corollary.

THEOREM 4: *Suppose that $\pi$ is a cuspidal automorphic representation of $U(F_\mathbb{A})$ such that $\pi_{v_0}$ is generic for every place $v_0$. Then $\pi$ is globally generic.*

*Proof:* Indeed, there is a cuspidal automorphic $\pi'$ in the same packet as $\pi$. By the previous result $\pi_{v_0} = \pi'_{v_0}$ for all $v_0$. Thus $\pi = \pi'$. ∎

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